$$\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$$

Why is this series interesting? We know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. The given series $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ is "between" these two series. This observation suggests that we use either $\sum_{k=1}^{\infty} \frac{1}{k^2}$ or $\sum_{k=1}^{\infty} \frac{1}{k}$ as a comparison series. In the first case, letting $a_k = \ln k/k^2$ and $b_k = 1/k^2$, we find that

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k/k^2}{1/k^2} = \lim_{k\to\infty} \ln k = \infty.$$

Case (3) of the Limit Comparison Test does not apply here because the comparison series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. So the test is inconclusive.

If, instead, we use the comparison series $\sum b_k = \sum \frac{1}{k}$, then

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k/k^2}{1/k} = \lim_{k\to\infty} \frac{\ln k}{k} = 0.$$

Case (2) of the Limit Comparison Test does not apply here because the comparison series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Again, the test is inconclusive.

With a bit more cunning, the Limit Comparison Test becomes conclusive. A series that lies "between" $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ is the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$; we try it as a comparison series. Letting $a_k = \ln k/k^2$ and $b_k = 1/k^{3/2}$, we find that

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k/k^2}{1/k^{3/2}} = \lim_{k\to\infty} \frac{\ln k}{\sqrt{k}} = 0.$$

(This limit is evaluated using l'Hôpital's Rule or by recalling that $\ln k$ grows more slowly than any positive power of k.) Now case (2) of the Limit Comparison Test applies; the comparison series $\sum \frac{1}{k^{3/2}}$ converges, so the given series converges.

Related Exercises 27–38◄