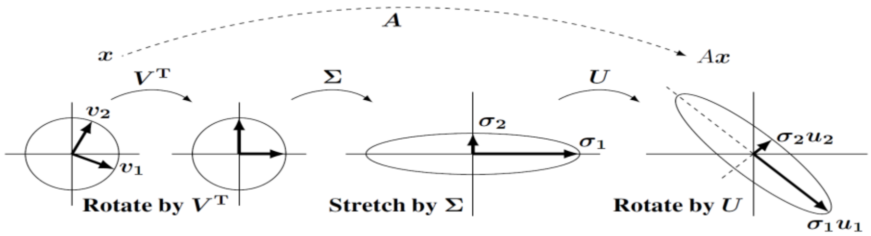
Virtual Mathematics Colloquium

Tuesday, October 13, 4:00 pm - 5:00 pm



The best picture related to these matrices shows the $SVD =$ Singular Value Decomposition of a matrix. $A =$ Rotation × Stretch × Rotation = Orthogonal × Diagonal × Orthogonal.

The Column-Row Factorization $A = CR$

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Abstract

In teaching linear algebra, I usually start with small matrices of integers. It is easy to see the independent and dependent columns of $\boldsymbol{A}$ (starting from column 1). The $\boldsymbol{r}$ independent columns go into a matrix $\boldsymbol{C}$. Then all columns are combinations of those columns, and this is expressed by $\boldsymbol{A} = \boldsymbol{CR} = (\boldsymbol{m} \times \boldsymbol{r})(\boldsymbol{r} \times \boldsymbol{n})$. The good thing is that $\boldsymbol{R}$ is exactly the row reduced echelon form of $\boldsymbol{A}$, with $\boldsymbol{m} - \boldsymbol{r}$ zero rows removed. So the row rank of $\boldsymbol{A}$ is also $\boldsymbol{r}$!

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \boldsymbol{CR}$$

This factorization makes clear the rank and the column space and the idea of independence and (most of all) the rule for multiplication: Columns of $\boldsymbol{CR}$ are combinations of columns of $\boldsymbol{C}$. The course is properly under way. And the first great theorem falls out: Column rank equals row rank. $\boldsymbol{R}$ has $\boldsymbol{r}$ independent rows and $\boldsymbol{A} = \boldsymbol{CR}$ expresses all rows of $\boldsymbol{A}$ as combinations of the rows of $\boldsymbol{R}$. There is a neat restatement of $\boldsymbol{CR}$ that treats rows and columns symmetrically. As in the example, suppose the first $\boldsymbol{r}$ columns and first $\boldsymbol{r}$ rows of $\boldsymbol{A}$ are independent. They meet in an $\boldsymbol{r} \times \boldsymbol{r}$ invertible matrix $\boldsymbol{W}$. Then $\boldsymbol{R}$ can be separated into $\boldsymbol{W}^{-1}$ times the submatrix $\boldsymbol{R}^*$ coming directly from $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{CW}^{-1}\boldsymbol{R}^* = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}.$$

Please click the link below to join the zoom webinar:
https://geneseo.zoom.us/j/92726687549?pwd=Vmo4R0RUV0E5RlJyNWtSNWVNSkRWQT09
Passcode: 467782