MATH 380-01– Integral equations and ill-posed problems, Spring 2019

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Integral equations

Integral equation are used in most applied areas and are as important as differential equations. In fact, many problems can be formulated (equivalently) as either a differential or an integral equation. So what is an integral equation?

Definition 1. *An integral equation is an equation containing an unknown function under the integral sign.*

Throughout the semester, one of our goals will be dedicated in approaches to determine that unknown function under the integral sign.

We will consider but will not be limited to the following equations.

- A linear **Fredholm integral equation of the first kind** has the form

$$\int_a^b K(x,y)f(y)\,dy = g(x), \tag{1}$$

  where K and g are known functions and f is the unknown function to be determined.

- A linear **Fredholm integral equation of the second kind** has the form

$$f(x) - \lambda \int_a^b K(x,y)f(y)\,dy = g(x),$$

  where K and g are known functions and f is the unknown function to be determined. Also, λ ($\in \mathbb{R}$ or $\mathbb{C}$) is a parameter.

- A **Volterra integral equation of the first kind** has the form

$$\int_a^x K(x,y)f(y)\,dy = g(x),$$

  where K and g are known functions and f is the unknown function to be determined.

- A **Volterra integral equation of the second kind** has the form

$$f(x) - \lambda \int_a^x K(x,y)f(y)\,dy = g(x),$$

  where K and g are known functions, f is the unknown function to be determined, and λ is a parameter.

Ill-posed problems

Definition 2 (**Well-posedness**)**.** *A problem is said to be* well-posed *(in the sense of Hadamard) if the following conditions hold*

1: existence: *there exists at least one solution.*

2: uniqueness: *the exists at most one solution.*

3: stability (also called continuous dependence): *the solution depends continuously on the given data.*

Definition 3 (**Ill-posedness**). *Problems in which one or more of the conditions of Definition 2 fails to hold are called* ill-posed *problems.*

The Fredholm integral equation of the first kind (1) is an example of an ill-posed problem.

A particular attention will be also given to **inverse problems**, which are special types of ill-posed problems. In short, *an inverse problem* is a problem that requires us to infer some cause or causes from effects that we can measure. For inverse problems, condition 2 and/or 3 of Definition 2 fail(s) to hold. We will use the well-known *Tikhonov regularization* to help combat ill-posedness in the approximate solutions.

Learning Outcomes

Upon successful completion of this course, a student will be able to:

- Reduce a given problem to an integral equation;
- Find solutions of the Volterra equation of the first kind;
- Solve the Fredholm equations of the first kind and of the second kind;
- Employ numerical methods to approximate solutions of integral equations;
- Use Tikhonov regularization method to combat ill-posedness in the solutions;
- Develop the aptitude to apply integral equation techniques to solve applied and/or theoretical problems such as computer tomography problems and many more.