ProofSpace Problem Set

Preliminaries

Introduction to Logic

This first problem set is unusual. Unlike all other problem sets, this one is mostly unrelated to the topics covered by the screencasts, as that content was covered well enough in the comprehension quiz. Instead of going deeper, we will take this opportunity at the beginning to discuss the importance of language and avoiding ambiguity, mathematical mistakes, and logical errors.

Discussed Problems

A **theorem** is a statement we can prove from previously agreed upon definitions, previously proven theorems, and logical reasoning. A **proposition** is used in professional mathematics to refer to a little theorem, and a **lemma** is a result that is proven on the way to proving a larger theorem. A **corollary** is an immediate consequence of a theorem. A **conjecture** is a hypothesis or guess that has not yet been proven or disproven.

In our problem sets and quizzes, stated propositions and theorems aren't necessarily true, as in the first problem below. In the screencasts and summary materials, however, they are always true.

1 Consider the following proof that 1=2, and identify the error(s) in the proof.

Theorem: 1=2.

Proof:

1) Suppose a and b are integers such that $a = b$.
2) Multiplying both sides by a, we get $a^2 = ab$.
3) Subtracting b^2, we get $a^2 - b^2 = ab - b^2$.
4) Then, we factor to get $(a - b)(a + b) = b(a - b)$.
5) Dividing both sides by $a - b$, we get that $a + b = b$.
6) Since $a = b$, it follows that $b + b = b$.
7) Thus, $2b = b$.
8) Dividing both sides by b, we conclude that $2 = 1$, as desired. **Q.E.D.**

2 The following problem is an adaptation of a problem from Edward Burger's *Extending the Frontiers of Mathematics: Inquiries into Proof and Augmentation.*

> **The Quest:** That famous archeologist Ohio Jones must save the cultural legacy of the great civilization of Oeseneg by recapturing its stolen precious icon, the Golden Bear. He finds two boxes labeled A and B, and he knows that each box contains the Gold Bear or deadly rattlesnakes. He must open one box, and the other box and its contents will be destroyed when the chosen box is opened. He also knows that either both of the following statements are true or both are false:
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> i) The Golden Bear is in one of the boxes;
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> ii) There are snakes in box A.
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> Can you give him advice as to which box to open? Remember: Safety first, but there may be more than one correct answer.

3 Decide if each logical deduction below is sensible or nonsensical. If a deduction feels "iffy," explain why. You may find it helpful to think about these from both a natural language perspective and a mathematical logic perspective.

a) To be a good basketball player, one must be athletic or work very hard. LeBron James works very hard. Therefore, he's not very athletic.

b) Today is Tuesday or Thursday. It is Thursday. Therefore, it is not Tuesday.

c) I will choose soup or salad. I will not choose soup. Therefore, I will choose salad.

The following are a preview of logical connectives that will be introduced in the following section.

d) If you are sick, then you skip classes. Steven skipped class. Therefore, he is sick.

e) If I live on campus, then I have a meal plan. I do not live on campus. Therefore, I do not have a meal plan.

f) If a person is born in Iowa, then he or she is a United States citizen. John Wayne was born in Iowa. Therefore, he was a U.S. citizen.

g) If I am President of the United States, then I can veto Congress. I am not President. Therefore, I cannot veto Congress.

h) If I have a cold or the flu, then I am sick. I don't have the flu. Therefore, I am not sick.

i) If I have a fever and a stomach ache, then I am sick. I have a fever. Therefore, I am sick.

Evaluated Problems

1 Consider the following proof that 1=2, and identify the error(s) in the proof.

Theorem: 1=2.

Proof:

1) Notice that $-2 = -2$.
2) We can rewrite this as $4 - 6 = 1 - 3$.
3) Add $\frac{9}{4}$ to both sides to get $4 - 6 + \frac{9}{4} = 1 - 3 + \frac{9}{4}$.
4) Factoring, we get $(2 - \frac{3}{2})^2 = (1 - \frac{3}{2})^2$.
5) Thus, $2 - \frac{3}{2} = 1 - \frac{3}{2}$.
6) We conclude that $2 = 1$, as desired. **Q.E.D.**

2 In mathematics, an **axiom** is something that we assume to be true. You may have seen axioms called "postulates" in a previous geometry course. We use axioms, or assumed truths, when proving statements in mathematics. Theorems may be consequences of a given collection of axioms, meaning that we can use the axioms to prove the theorem. The rules in "The Quest" problem above could be called axioms.

Here is a set of axioms that define how a farmer must grow his crops.

Axiom 1. Each year, the farmer must grow exactly three of the following crops:

beans, corn, kale, peas, and squash.

Axiom 2. If the farmer grows corn, he must grow beans as well.
Axiom 3. If the farmer grows peas, he cannot grow kale.
Axiom 4. If the farmer grows kale one year, he cannot grow kale the next year.
Axiom 5. If the farmer grows corn one year, he must grow peas the next year.
Axiom 6. The farmer must grow exactly one of kale or squash each year.

Based on these axioms, prove the following theorems.

Theorem 1: If the farmer grows peas, he cannot grow corn.
Theorem 2: If the farmer grows kale one year, he must grow peas the next year.
Theorem 3: The farmer must grow beans.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 1.1: 1

Advanced Problems

In general, clever and careful approaches are going to be the most successful in your excursion to proofs.

1 The following problem was drawn from Martin Gardner's *My Best Mathematical and Logical Puzzles.*

(The Mutilated Chessboard) "The props for this problem are a chessboard and 32 dominoes. Each domino is of such size that it exactly covers two adjacent squares on the board. The 32 dominoes therefore can cover all 64 of the chessboard squares. But now suppose we cut off two squares at diagonally opposite corners of the board and discard one of the dominoes. Is it possible to place the 31 dominoes on the board so that all the remaining 62 squares are covered? If so, show how it can be done. If not, prove it impossible." HINT: Consider the checkered pattern of a chessboard and the colors of the squares.

2 The following problem was drawn from Edward Burger and Michael Starbird's *The Heart of Mathematics.*

(That's a Meanie Genie) On an archaeological dig near the highlands of Tibet, Alley discovered an ancient oil lamp. Just for laughs she rubbed the lamp. She quickly stopped laughing when a huge puff of magenta smoke spouted from the lamp, and an ornery genie named Murray appeared. Murray, looking at the stunned Alley, exclaimed, "Well, what are you staring at? Okay, okay, you've found me; you get your three wishes. So, what will they be?" Alley, although in shock, realized she had an incredible opportunity. Thinking quickly, she said, "I'd like to find the Rama Nujan, the jewel that was first discovered by Hardy the High Lama." "You got it," replied Murray, and instantly nine identical-looking stones appeared. Alley looked at the stones and was unable to differentiate any one from the others.

Finally she said to Murray, "So where is the Rama Nujan?" Murray explained, "It is embedded in one of these stones. You said you wished to find it. So now you get to find it. Oh, by the way, you may take only one of the stones with you, so choose wisely!" "But they look identical to me. How will I know which one has the Rama Nujan in it?" Alley questioned. "Well, eight of the stones weigh the same, but the stone containing the jewel weighs slightly more than the others," Murray responded with a devilish grin.

Alley, becoming annoyed, whispered under her breath, "Gee, I wish I had a balance scale." Suddenly a balance scale appeared. "That was wish two!" declared Murray. "Hey, that's not fair!" Alley cried. "You want to talk fair? You think it's fair to be locked in a lamp for 1729 years? You know you can't get cable TV in there, and there's no room for a satellite dish! So don't talk to me about fair," Murray exclaimed. Realizing he had gone a bit overboard, Murray proclaimed, "Hey, I want to help you out, so let me give you a tip: That balance scale may be used only once." "What? Only once? she said, thinking out loud. "I wish I had another balance scale." ZAP! Another scale appeared. "Okay, kiddo, that was wish three," Murray snickered. "Hey, just one minute," Alley said, now regretting not having asked for one million dollars or something more standard. "Well at least this new scale works correctly, right?" "Sure, just like the other one. You may use it only once." "Why?" Alley inquired. "Because it is a 'wished' balance scale," he said, so the rule is 'one scale, one balancing'; it's just like the rule against using one wish to wish for a hundred more wishes." "You are a very obnoxious genie." "Hey, I don't make up the rules, lady, I just follow them," he said.

So, Alley may use each of the two balance scales exactly once. Is it possible for Alley to select the slightly heavier stone containing the Rama Nujan from among the nine identical-looking stones? Explain why or why not.

3 The following example of an axiom system is attributed to Steven Lay's *Analysis with an Introduction to Proof.*

> The math club at Podunk University has a number of committees according to the following rules established in its constitution:
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> i) There must be at least one committee.
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> ii) Each committee has at least three members.
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> iii) Any two distinct committees have exactly one member in common.
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> iv) For each pair of members of the club, there is one and only one committee of which both are members.
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> v) Given any committee, there exists at least one member of the club who is not a member of that committee.

a) Prove that there exist at least three members in the club.

b) Prove that there exist at least three committees.

c) Prove that if x is a member of the club, then there is at least one committee of which x is not a member.

d) Prove that every member of the club is a member of at least three committess.

e) Prove that there are at least seven members in the club.

f) Prove that there are at least seven committees.

g) Describe a club with seven members and seven committees that meets all the required conditions.

4 Here is another problem involving axioms. It is attributed to Steven Lay's *Analysis with an Introduction to Proof*.

> The math club at Popular College has a number of committees according to the following rules established in its constitution:
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> i) Every member of the club is a member of at least one committee.
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> ii) For each pair of members of the club, there is one and only one committee of which both are members.
>
> iii) For every committee, there is one and only one committee that has no members in common with it.
>
> iv) Every committee must contain at least one member of the club.

a) Prove that every member of the club is a member of at least two committees.

b) Prove that if the club has at least one member, then it has at least four members.

c) Suppose that the club has exactly four members. What is the minimum number of committees? Why?