ProofSpace Problem Set

Preliminaries

Conditional Statements

Discussed Problems

1 Determine the truth values of P, Q, and R if we assume the following three compound statements are all true:

(i) $R \Rightarrow Q$ (ii) $\neg P \wedge Q$ (iii) $\neg(R \Rightarrow \neg Q)$

2 Suppose P and Q are false and R is true. What is the truth value of each of the following?

a) $\neg P \Rightarrow (Q \Rightarrow (P \wedge R))$

b) $((R \vee Q) \Rightarrow P) \wedge R$

c) $(P \vee R) \Rightarrow (\neg Q \vee R)$

3 Let x be any real number. Determine the values of x (if there are any) that make the following statements true.

a) If $4 < 6$, then $x + 3 = 8$.

b) If $6 < 4$, then $x + 3 = 8$.

c) If $x + 3 = 8$, then $6 < 4$.

4 Let x be any real number. Consider the following theorem:

Theorem: Let a, b and c be real numbers and $a \neq 0$. If f is a polynomial of the form $f(x) = ax^2 + bx + c$ and $b^2 - 4ac > 0$, then the function has exactly two $x-$intercepts.

Using just the theorem above, what can you conclude about the following functions?

a) $f(x) = 3x^2 + 6x + 3$

b) $g(x) = -x^2 + 2x + 2$

c) $h(x) = 4x^3 + 6x^2 + 2x + 5$

d) $j(x) = 2x + 5$

e) $k(x) = 5x^2 - 4x$

f) $s(x) = 5x^2 - 4x + 1$

Evaluated Problems

1 Prove that the following statement, called *modus ponens*, is a tautology.

$$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$$

2 Suppose P and R are false and Q is true. What is the truth value of each of the following?

a) $P \vee (Q \Rightarrow (P \wedge R))$

b) $((R \vee Q) \iff P) \wedge P$

c) $(P \vee R) \Rightarrow (\neg Q \wedge R)$

3 (Sundstrom) Let x be any real number. Consider the following theorems:

> Let b be a real number.
>
> **Theorem A:** If f is a cubic function of the form $f(x) = x^3 - x + b$ and $b > 1$, then the function f has exactly one $x-$intercept.
>
> **Theorem B:** If f and g are functions with $g(x) = k \cdot f(x)$, where k is a nonzero real number, then f and g have exactly the same $x-$ intercepts.

Using just the theorems above, what can you conclude about the following functions? (Does the function have the appropriate form for you to conclude that it has exactly one $x-$intercept? Is it a nonzero multiple of a function of the appropriate form?)

a) $f(x) = x^3 - x + 7$

b) $g(x) = x^3 + x + 7$

c) $k(x) = -x^3 + x - 5$

d) $h(x) = -x^2 + x - 5$

e) $F(x) = 2x^3 + wx + 6$, for some integer w

f) $G(x) = x^4 - x + 11$

g) $H(x) = 2x^3 - 2x + 7$

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 1.1: 3, 4, 5, 6

Advanced Problems

1 The following line of reasoning is extrapolated from Raymond Smullyan's book, *What Is the Name of This Book?*. It provides an introduction to the field called "Mathematical Logic."

a) Consider the statement

> This sentence can never be proved.

Use the fact that we can only prove true sentences to derive a contradiction.

b) The reason the sentence above leads to contradiction is because the notion of *provable* is ill-defined. We usually speak of an **axiomatic system** $\mathfrak{M}$, which is a series of *axioms* (assumed truths). In a proof, we are allowed to use the axioms from $\mathfrak{M}$. Therefore, we can only speak practically about a given statement P being provable or not provable *within a system* $\mathfrak{M}$.

We call a system $\mathfrak{M}$ **sound** provided that in $\mathfrak{M}$, if a statement is provable, it is true. That is, if a statement is false, we cannot prove it in $\mathfrak{M}$.

Suppose we have a sound axiomatic system $\mathfrak{B}$. Consider the following sentence:

> This sentence cannot be proved in the system $\mathfrak{B}$.

Prove that the sentence is true.

c) We call a system $\mathfrak{M}$ **complete** provided that in $\mathfrak{M}$, if a statement is true, then it can be proven. That is, if a statement is true, we can prove it in $\mathfrak{M}$.

Notice that the implication used in the definition of complete is the converse of the implication used in the definition of sound. Obviously the ideal system is **sound and complete** - everything that is true can be proven, and everything that can be proven is true. You may be saddened to find out that our usual mathematics, that allows basic addition and multiplication, is **incomplete** (not complete).

Recall that the statement

> This sentence cannot be proved in the system $\mathfrak{B}$.

is true. Prove that $\mathfrak{B}$ is incomplete.

The notion of "true" used in this problem is a little bit vague. For the purposes of virtually every non-logic math course you take, your working definition of truth will suffice. However, there is a "rigorous" definition of truth used by math logicians.