ProofSpace Problem Set

Proof Techniques

Contrapositive and Contradiction

Discussed Problems

For each of the following, prove or disprove. If a biconditional is false, prove or disprove each direction. Consider whether a direct proof, indirect proof, or neither is recommended for each problem.

1) No integer is both even and odd. (Hint: Negative statements such as this can often be stated in the form of a conditional. Can you think of a way to express this as a conditional statement?)
2) Let a and b be integers. If a is even and b is odd, then $4 \nmid (a^2 + 2b^2)$.
3) For every positive rational number q, there exists a positive rational number p such that $p < q$. (Hint: Consider a technique known as a *constructive proof.*)
4) There do not exist integers a and b such that $9a + 15b = 5$.
5) Let n be an integer. Then n is odd if and only if n^3 is odd.
6) Let x be a positive real number. Then x is irrational if and only if $\sqrt{x}$ is irrational.

Evaluated Problems

For each of the following, prove or disprove. If a biconditional is false, prove or disprove each direction.

1) Let x and y be real numbers. If x is rational and y is irrational, then $x + y$ is irrational.
2) There do not exist integers a and b such that $9a + 15b = 6$.
3) Let a be an integer. Then, a is even if and only if $4 \mid a^2$.
4) Let x and y be real. If xy is irrational, then x is irrational or y is irrational.
5) For all integers a and b, if $a \not\equiv 0 (\text{mod } 6)$ and $b \not\equiv 0 (\text{mod } 6)$, then $ab \not\equiv 0 (\text{mod } 6)$.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 3.2: 1, 2, 6, 9, 13, 19
Sec. 3.3: 2, 3, 4, 6, 13, 20

Advanced Problems

1 One of the most famous proofs by contradiction is Euclid's famous proof that there are infinitely many primes.

> **Theorem:** Every natural number greater than one has at least one prime factor greater than 1.

This theorem can actually be made much stronger, but we state it because it is necessary for Euclid's proof.

> **Theorem:** There are infinitely many primes.
>
> **Proof:** (Euclid) Assume for the sake of contradiction that there are finitely many primes, $p_1, p_2, ...p_n$. Let N be their product plus one, that is $N = p_1p_2...p_n + 1$. Since every natural number greater than one has at least one prime divisor, there exists a prime number p such that $p \mid N$.

Complete Euclid's proof. HINT: Suppose $a \mid b$ and $a \mid b + 1$. What can we conclude?