ProofSpace Problem Set

Proof Techniques

Induction

Discussed Problems

1 Let n be a natural number. Let $Q(n)$ be the statement

$$2 + 4 + 6 + ... + 2n = n(n + 1).$$

a) What is $Q(1)$? $Q(2)$? $Q(4)$? $Q(n + 1)$? $Q(2n)$?

b) Prove $Q(n)$ for all natural numbers.

2 Prove the following statement for all natural numbers:

$$1^3 + 2^3 + ... + n^3 = \left[\frac{n(n + 1)}{2}\right]^2.$$

3 Prove that every natural number is either even or odd.

4 (Tower of Hanoi Puzzle) Consider a board with three pegs. On the first peg, there is a tower of three disks: one with radius 3 in. on the bottom, radius 2 in. above it, and radius 1 in. above that. Our goal is to move the tower, in size order, onto the third peg, with the following restrictions:

i) Only one disk may be moved at a time.

ii) A disk can only be moved if it is the top disk in a stack.

iii) No disk may be placed on top of a disk with a smaller radius.

Suppose a "move" is any movement of a single disk off of one peg and onto another.

(a) Explain why it takes at least 7 moves to complete the puzzle with three disks.

(b) Suppose the puzzle began with only two disks. What is the smallest number of moves necessary to complete the puzzle?

(c) Suppose the puzzle began with four disks. Use the fact that it takes at least 7 moves to complete the three disk puzzle to verify that it takes at least 15 moves to complete the four disk puzzle.

(d) Make a conjecture about the smallest number of moves it takes to complete the puzzle with 5 disks.

(e) Make a conjecture about the smallest number of moves it takes to complete the puzzle with n disks.

> "Let $n \in \mathbb{N}$. If n disks are used in the Towers of Hanoi puzzle, it will take at least ____________ moves to complete the puzzle."

Evaluated Problems

1 Let n be a natural number. Let $Q(n)$ be the statement $1+5+9+...+(4n-3) = n(2n-1)$.

a) What is $Q(1)$? $Q(4)$? $Q(n+1)$? $Q(2n)$?

b) Prove $Q(n)$ for all natural numbers.

2 Prove the following statement for all natural numbers:

$$1 + 2(2) + 3(2^2) + ... + n(2^{n-1}) = (n-1)2^n + 1.$$

3 Use induction to prove that for all natural numbers n, $3 \mid (n^3 + 23n)$.
(We used cases to prove this in the Discussed Problems of Section 2.3. Induction allows us to prove this *without* using cases.)

4 Prove your conjecture from Discussed Problem 4(e) above.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 4.1: 3, 5, 8, 13, 18

Advanced

1 (Cohen, adapted) In a previous problem set, we foolishly tried to prove that unicorns exist. We have since moved on to more earthly propositions. Consider the following proof that all students have the same major.

Proof: We will prove by induction that any group of n students have the same major, thus proving that all students have the same major. Clearly a set with just 1 student is a set of students with the same major, so the proposition holds for the base case $n = 1$.

Let k be a natural number, and assume that a group of k students consists entirely of students with the same major.

Consider a group of $k + 1$ students, numbered 1 through $k + 1$. Remove the first student, leaving a group (call it Group One) of k students. By our induction hypothesis, Group One consists entirely of students with the same major. Now, return to our original group and remove the second student, creating a group of k students called Group Two. Again, Group Two consists entirely of students with the same major. Since our original group consists of all the students from Group One and all the students from Group Two, all the students in our original group must have the same major.

Therefore, it is true by induction that all students have the same major. $\square$.

Phew! That proof was hard work. Wait, hold on. We've just been informed that **not** all students have the same major. You might even know someone with a different major (we hope you do!). We'll try not to let this minor setback get us down.

Ignoring stylistic issues, what was wrong with our proof?

2 (Base Cases) Consider a proof by induction for a proposition $P(n)$.

a) Suppose that in a proof, the writer proved $P(2)$ as the base case instead of $P(1)$. For which numbers did the writer prove the proposition?

b) Suppose that the writer proved $P(146)$ as the base case instead of $P(1)$. For which numbers did the writer prove the proposition?

c) Suppose that the writer assumed as her induction hypothesis that $P(r - 1)$ was true and used it to show $P(r)$ was true. Is this a valid proof by induction on $\mathbb{N}$? Why or why not?