ProofSpace Problem Set

Sets

Proving Subsets and Set Equality

Discussed Problems

If a biconditional or equality is false, determine if either of the directions or containments is true and prove/disprove all claims.

1 Prove or Disprove: Let A and B be sets. If $A \nsubseteq B$ and $B \nsubseteq A$, then A and B are disjoint.

2 For each of the following pairs of sets, decide if the sets are equal. If not, is one a subset of the other? Prove your results.

a) Let $A = \{x \in \mathbb{R} \mid (\forall y \in \mathbb{R})(y < 0 \Rightarrow xy > 0)\}$.
Let $B = \{x \in \mathbb{R} \mid (\nexists y \in \mathbb{R})(e^y = x)\}$.

b) Let $C = \{x \in \mathbb{Z} \mid 4 \text{ divides } x\}$.
Let $D = \{x \in \mathbb{Z} \mid (\exists y \in \mathbb{Z})(\exists z \in \mathbb{Z})(x = 2y + 2z)\}$.

3 Let $E = \{x \in \mathbb{Z} \mid 3 \text{ divides } x\}$. Let $F = \{x \in \mathbb{Z} \mid (\exists p \in \mathbb{Z})(x^2 + x - 3p + 7 = 0)\}$. Prove that E and F are disjoint.

4 Let A, B, and C be arbitrary sets. In one of the screencasts, we proved one of the distributive laws: $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. Now prove the other distributive law:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

5 Do there exist sets A and B such that $A = B$ and A and B are disjoint? If so, what are they? If not, why not?

Evaluated Problems

If a biconditional or equality is false, determine if either of the directions or containments is true and prove/disprove all claims.

1 Prove or Disprove: Let A and B be sets. A and B are disjoint if and only if for every $x \in A$, $x \notin B$.

2 For each of the following pairs of sets, decide if the sets are equal. If not, is one a subset of the other? Prove your results.

a) Let $A = \{x \in \mathbb{Q} \mid x^2 \leq 4\}$. Let $B = \{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$.

b) Let $C = \{x \in \mathbb{Z} \mid 5 \text{ divides } x\}$. Let $D = \{x \in \mathbb{Z} \mid (\exists y \in \mathbb{Z})(\exists z \in \mathbb{Z})(x = 10y + 15z)\}$.

3 Let A and B be sets. Prove or Disprove:$A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

4 Let A, B, C, and D be sets such that $A \subseteq B$ and $C \subseteq D$. Prove or Disprove: B and D are disjoint if and only if A and C are disjoint.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 5.1: 1, 2, 4, 5, 6, 13

Advanced Problems

1 In this problem, we will discuss Russell's Paradox, a major problem in naive (old) set theory:

a) Consider the set $H = \{1, 2, 3\}$. Is $H \in H$?

b) Consider the set $E = \{x \mid x \text{ is an isoceles triangle}\}$. Is $E \in E$?

c) Consider the set $L = \{x \mid x \text{ is a set}\}$. Is $L \in L$?

d) Consider the set $P = \{P\}$. Is $P \in P$?

e) Let $A = \{B \mid B \text{ is a set and } B \notin B\}$. Which of H, E, L and P above are elements of A?

f) Is $A \in A$? Why or why not?

This contradiction is referred to as "Russell's Paradox." As a consequence, we have added the "axiom of regularity" to our axioms of set theory. This axiom asserts that no set can be an element of itself. Thus, any "set" above that you said was an element of itself was not a set at all.

The paradox is often explained as the "barber paradox." Suppose in a town, the law says that every man must shave every day. A man needn't shave himself, though. The town, probably because they felt bad about creating such a ludicrous law, provides the men a barber. The rule is that a man is shaved by the barber if and only if he does not shave himself. The question is this: who shaves the barber?

2 Prove that there is only one empty set.