ProofSpace Problem Set

Sets

Operations and Identities

Discussed Problems

This is by no means an exhaustive list of questions pertaining to this week's material. We strongly recommend checking out some of the supplementary problems if you're having trouble.

For each of the following, prove or disprove. If an equality is false, prove or disprove each of the containments. Use prior knowledge whenever possible. Let U be the universal set, and let A, B, and C be arbitrary sets.

1. (DeMorgan's Law 1) $(A \cup B)^c = A^c \cap B^c$.
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
3. $(A \times B) \cup C = (A \cup C) \times (B \cup C)$.
4. $A - (A \cap B^c) = A \cap B$.
5. $(A - B)^c = (A \cup B)^c \cup B$.
6. $B - A$ and $A - B$ are disjoint.

Evaluated Problems

For each of the following, prove or disprove. If a biconditional or equality is false, determine if either of the directions or containments is true and prove/disprove all claims. Use prior knowledge whenever possible. Let U be the universal set, and let A, B, and C be arbitrary sets.

1. (DeMorgan's Law 2) $(A \cap B)^c = A^c \cup B^c$.
2. $A \subseteq B$ if and only if $B^c \subseteq A^c$.
3. If $Y \subseteq A$, then $Y \times B \subseteq A \times B$.
4. (a) Suppose A and B are nonempty. Then $A \times B = B \times A$ if and only if $A = B$.
   (b) Is the nonempty restriction required for the biconditional to be true? Prove your claim. Also, if it is required, indicate where you used this in your proof of part (a).

5. $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
6. $A - B$ and $A \cap B$ are disjoint sets.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 5.3: 3, 5, 7, 10, 11, 12.
Sec. 5.4: 1, 2, 3, 5, 8.

Advanced Problems

1 (Ross-Littlewood Paradox, adapted) In this problem, we will look at the strange and fascinating nature of infinity, with which many set theorists are rightfully obsessed.

a) In the Kingdom of Foreverland, the King and Queen arrive at the royal treasury at 11 a.m. The King put two gold coins on the ground, and the Queen picked up the top one. 30 minutes before noon, the King again put down two gold coins, and the Queen again picked up the top one. This occurred again 15 minutes before noon, then 7.5 before noon, 3.25, etc... until noon itself. The King and Queen move very fast and could find a way to do this - don't try it at home. How many coins were in the pile at noon?

b) In the Kingdom of Forevereverland, the King and Queen arrive at the royal treasury at 11 a.m. The Queen put two gold coins on the ground, and the King picked up the bottom one. 30 minutes before noon, the Queen again put down two gold coins, and the King again picked up the bottom one. This occurred again 15 minutes before noon, then 7.5 before noon, 3.25, etc... until noon itself. This King and Queen are also spectacularly fast, and the King found a way to do this without toppling the pile. Ah, to be royal. How many coins were in the pile at noon?

Though possibly perplexing, this deals with a philosophical notion called "supertasks," (in which an infinite number of tasks are done in a finite amount of time) and will not affect the development of our rich theory of infinity.

2 We defined an ordered pair to be a pair of objects in a specified order, with the property that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. In this problem, we will prove this property.

a) Define (a, b) to be the set $\{\{a\}, \{a, b\}\}$. This allows us to express an ordered pair as a set. For example, $(1, 2)$ would be $\{\{1\}, \{1, 2\}\}$. Express $(2, 1)$ and $(1, 1)$ as sets.

b) Prove using the set theoretic definition of ordered pair that if $a = c$ and $b = d$, then $(a, b) = (c, d)$.

c) Suppose $(a, b) = (c, d)$. Then, $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Suppose $a = b$. Show then that $a = c$, and therefore that $b = d$.

d) Now, suppose that $a \neq b$. Prove that $a = c$ and that $b = d$.

There are many other set theoretic ways to define an ordered pair. A good definition is one for which we can prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.