ProofSpace Problem Set

Functions

Composition and Inverses

Discussed Problems

Let A, B, C, and D be arbitrary sets.

1 Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Prove that $h \circ (g \circ f) = (h \circ g) \circ f$.

2 Let $f : A \to B$ and $g : B \to C$. Prove or disprove:

(a) If $g \circ f$ is a surjection, then g is a surjection.

(b) If g is a surjection, then $g \circ f$ is a surjection.

(c) If $g \circ f$ is an injection, then g is an injection.

(d) If g is an injection, then $g \circ f$ is an injection.

3 (Iterated Functions) Let $f : A \to A$. For all $n \in \mathbb{N}$, define $f^1(x) = f(x)$ and if $n > 1$, define $f^n(x) = f(f^{n-1}(x))$.

(a) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 2x + 1$. Find $g^n(x)$ for $1 \leq n \leq 4$.

(b) Make a conjecture for a formula for $g^n(x)$.

4 Let $f : A \to B$ be a bijection. Prove that for all $x \in A$, $(f^{-1} \circ f)(x) = x$.

5 Find the inverse functions of the following functions. If the function is not invertible, explain why.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1-2x}{3}$.

(b) $g : \mathbb{Z} \to \mathbb{R}$ defined by $g(x) = \frac{1-2x}{3}$.

(c) $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $h(x, y) = (2y, x^3)$.

6 Decide if the following statement is true or false and explain why: If $f : A \to B$ is a bijection and $a \in B$, then $f^{-1}(a) = f^{-1}[\{a\}]$.

Evaluated Problems

Let A, B, and C be arbitrary sets.

1 Suppose $f : A \to B$ is a bijection. Prove that f^{-1} is a bijection without using the result of Problem 4 below. You are allowed to use the result of Discussion Problem 4.

2 Let $f : A \to B$ and $g : B \to C$. Prove or disprove:

(a) If $g \circ f$ is a surjection, then f is a surjection.

(b) If f is a surjection, then $g \circ f$ is a surjection.

(c) If $g \circ f$ is an injection, then f is an injection.

(d) If f is an injection, then $g \circ f$ is an injection.

3 (Iterated Functions) Prove your conjecture from Discussed Problem 3(b) above.
HINT: How do you prove something for all $n \in \mathbb{N}$?

4 Let $f : A \to B$ be a bijection. Prove that for all $y \in B$, $(f \circ f^{-1})(y) = y$.

5 Find the inverse functions of the following functions. If the function is not invertible, explain why.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x^3+1}{3}$.

(b) $g : \mathbb{Z} \to \mathbb{R}$ defined by $g(x) = \frac{x^3+1}{3}$.

(c) $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $h(x, y) = (2x + y, x + y)$.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 6.4: 3, 5a, 10.
Sec. 6.5: 3(a,b), 4, 6, 7, 8, 9

Advanced Problems

1 Let A and B be sets with more than one element. The function $proj_A : A \times B \to A$ is given by $proj_A(x, y) = x$.

a) Does there exist a function $h : A \to A \times B$ such that $h(proj_A(x, y)) = (x, y)$? If so, what is it? If not, why not?

b) What happens if B does not have more than one element?

2 Suppose $f : A \to B$ and $g : B \to C$ are bijections. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

3 In a previous problem (Sets/Indexed Sets/Advanced/1), we thought about $\prod_{i \in I} A_i$. We ran into some trouble when $I = \mathbb{R}$. In this problem, we will address that issue.

a) Explain why it makes sense to think of $\prod_{i \in I} A_i$ as a function $\prod : I \to \bigcup_{i \in I} A_i$.

b) With this definition, $\prod_{i \in I} A_i \subseteq \{f \mid f : I \to \bigcup_{i \in I} A_i\}$. Can you add an extra condition to the set $\{f \mid f : I \to \bigcup_{i \in I} A_i\}$ to establish equality?