ProofSpace Problem Set

Functions

Cardinality

Discussed Problems

Definition: Let S be a set. Then the *cardinality of* S, denoted $|S|$, describes how many elements are in S. If S is a finite set, then $|S|$ is the number of elements in the set. If S is an infinite set, then $|S| = \infty$.

Definition: Let S be a set such that $|S| = |\mathbb{N}|$. Then S is *countably infinite*, and we say "the cardinality of S is *aleph-naught*," and write $|S| = \aleph_0$. (This symbol is the first letter of the Hebrew alphabet, aleph, with a subscript zero).

Some of these problems are relatively difficult. If you are struggling with some new sets and definitions, write out some examples!

1 Let A, B, C, and D be sets such that $A \subseteq B$ and such that C and D are finite, with $|C| = m$ and $|D| = n$.

a) Prove that $|A| \leq |B|$.

b) Suppose $|A| = |B|$. Is it true that $A = B$?

c) Suppose C and D are disjoint. Explain why $|C \cup D| = m + n$.

d) Suppose C and D are not disjoint with $|C \cap D| = r$. Explain why $|C \cup D| = m + n - r$.

e) Prove that $|\mathbb{N} \cup \{0\}| = \aleph_0$.

f) Suppose E is a set such that $|E| = \aleph_0$. Prove that $|E \cup C| = \aleph_0$.

g) Prove that $|\mathbb{Z}| = \aleph_0$.

2 In this problem, we will consider the cardinalities of Cartesian products.

a) Prove that $|\mathbb{N} \times \mathbb{N}| = \aleph_0$.

b) Prove that for all $n \in \mathbb{N}$, $|\mathbb{N}^n| = \aleph_0$.

3 Let $2^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \{0, 1\}\}$.

a) Give four elements of $2^{\mathbb{N}}$.

b) Use Cantor's Diagonlization argument to prove that $2^{\mathbb{N}}$ is uncountable.

Evaluated Problems

1 Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Prove that $|\mathbb{R}| = |\mathbb{R}^+|$.

2 Let $\mathbb{N}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \mathbb{N}\}$.

a) Give two elements of $\mathbb{N}^{\mathbb{N}}$.

b) Prove that $\aleph_0 < |\mathbb{N}^{\mathbb{N}}|$.

3 Let A and B be sets such that $|A| = |B| = \aleph_0$. Prove or disprove:

a) $|A \cap B| = \aleph_0$.

b) $|A \cup B| = \aleph_0$.
(Hint: Consider the two cases (1) A and B are disjoint, and (2) A and B are not disjoint. In the second case, let $C = B - A$ and notice that $A \cup B = A \cup C$ and $A \cap C = \varnothing$. C could be finite or infinite.)

c) The irrational numbers have cardinality $\aleph_0$.
(Hint: Consider using part (b). Also, whether you decide to prove or disprove this statement, a formal proof is required.)

d) Let x be any number. $|A \times \{x\}| = \aleph_0$.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
9.1: 4, 5, 7
9.2: 1, 2, 7, 8, 9, 10, 13
9.3: 1, 7, 8 (Note that the book uses **c** to be the cardinality of $\mathbb{R}$)

Advanced Problems

1 In this problem, we will consider how many sets are countably infinite. Let A be any set. Let $B = \{A \mid |A| = \aleph_0\}$.

a) Recall Russell's Paradox (Sets/Proving Set Relationships/Advanced/1). Use Russell's Paradox to prove that B is uncountable.

b) Find an alternate proof that B is uncountable.

2 Let A be any set. Define $2^A = \{f \mid f : A \to \{0,1\}\}$. In this question, we will prove that $|2^A| = |\mathcal{P}(A)|$. Let $B = \{1,2,3\}$.

a) Find 2^B and find $|2^B|$.

b) Prove that $|2^B| = |\mathcal{P}(B)|$.

c) Why does it make sense to think about elements of 2^B as lists of 0's and 1's? Can you generalize to 2^A?

d) Define the characteristic function $\chi_A : 2^A \to \mathcal{P}(A)$ as follows: Let $g \in 2^A$. Then, for all $x \in A$, if $g(x) = 1$, then $x \in \chi_A(g)$ and if $g(x) = 0$, then $x \notin \chi_A(g)$.

   Let $h(x) : \mathbb{N} \to \{0,1\}$ be given by $h(x) = 1$ if and only if $3 \mid x$. What is $\chi_{\mathbb{N}}(h)$?

e) Prove that for any set A, $\chi_A : 2^A \to \mathcal{P}(A)$ is a bijection.

f) Use Cantor's Theorem to prove that for all sets A, $|A| < |2^A|$.

3 In this problem, we will define a series of cardinalities recursively for all $n \in \mathbb{N}$. In set theory, the index for the definitions provided below are called the *ordinals*, a series of sets that are an extension of the natural numbers.

a) Recall that there are multiple infinities. Define $\aleph_0$ as the cardinality of $\mathbb{N}$. For each $i+1 \in \mathbb{N}$, define $\aleph_{i+1}$ as the cardinality such that there are no sets A with the property that $\aleph_i < |A| < \aleph_{i+1}$. State the continuum hypothesis in terms of $\mathbb{R}$, $\aleph_0$, and $\aleph_1$.

b) The symbol $\beth$ is the second letter of the Hebrew alphabet and is called "beth." Define $\beth_0$ to be $\aleph_0$. For each $i+1 \in \mathbb{N}$, define $\beth_{i+1} = |2^{\beth_i}|$. Find a set with cardinality $\beth_1$.

c) State the continuum hypothesis in terms of $\aleph_i$ and $\beth_i$.

A modified version of the "generalized continuum hypothesis" states that for each $i \in \mathbb{N}$, $\aleph_i = \beth_i$ (the unmodified version states that is is true for any ordinal i).