ProofSpace Problem Set

Relations

Equivalence Classes

Discussed Problems

1 (Sundstrom) Consider the following relation on $\{x \in \mathbb{Z} \mid 0 < x \leq 1000\}$.

> $a \sim b$ if and only if a and b have the same number of digits.

This is an equivalence relation. What are the equivalence classes? (Provide a representative element $[x]$ and list the elements for each equivalence class.)

2 (Sundstrom) Consider the following relation $\sim$ on $\mathbb{R} \times \mathbb{R}$.

> $(a, b) \sim (c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$.

This is an equivalence relation.

a) List four elements of $[(0, 5)]$.

b) What is $[(0, 0)]$?

c) Without using the $\sim$ symbol, write out in set notation $[(2, 3)]$.

d) Describe $[(2, 3)]$ geometrically.

e) Can all equivalence classes be described geometrically in the same way? Why or why not?

3 Recall that if R is an equivalence relation and $[a]$ and $[b]$ are equivalence classes, then $[a] = [b]$ or $[a] \cap [b] = \emptyset$. Prove this fact.

4 For each of the following, define an equivalence relation R on $\mathbb{Z}$ with the given property. Be prepared to justify your answer. Your answer for d) may be related to your earlier answers, but does not have to be.

a) R has exactly one equivalence class.

b) R has exactly two equivalence classes.

c) R has exactly three equivalence classes.

d) Let $n \in \mathbb{N}$. R has exactly n equivalence classes.

Evaluated Problems

1 Consider the following equivalence relation S on $\mathbb{R}$.

$(a, b) \in S$ if and only if $(a - b) \in \mathbb{Q}$.

Provide four distinct equivalence classes of S, including a representative element $[x]$ and several elements of each equivalence class.

2 Recall Problem 2 above concerning the relation $\sim$ on $\mathbb{R} \times \mathbb{R}$.

$(a, b) \sim (c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$.

a) Prove $\sim$ is an equivalence relation.

b) Let $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$. Let A be the set of all equivalence classes of $\sim$. Find a bijection from $\mathbb{R}^*$ to A. Prove the function is injective.

3 Recall that if R is an equivalence relation and $[x]$ and $[y]$ are equivalence classes, then xRy if and only if $[x] = [y]$. Prove this fact.

4 For each of the following, define an equivalence relation R on $\mathbb{R}$ with the given property. Be prepared to justify your answer. Your answer for d) may be related to your earlier answers, but does not have to be.

a) R has exactly two equivalence classes.

b) R has exactly three equivalence classes.

c) R has exactly four equivalence classes.

d) Let $n \in \mathbb{N}$. R has exactly n equivalence classes.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 7.3: 1, 3, 4, 5, 9, 7, 10, 11.

Advanced Problems

1 Recall that for $x \in \mathbb{R}$, $|x|$, called the absolute value of x, is given by $\sqrt{x^2}$. In this problem, we will consider the way equivalence classes work in the mathematical field called **topology** by considering some equivalence relations.

a) Let X be the unit square region $\{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \text{ and } |y| \leq 1\}$. We can assign an equivalence relation to X so that two points (x_1, y_1) and (x_2, y_2) in X are equivalent if:
- they are the same point, or
- $x_1 = x_2$ and $|y_1| = |y_2| = 1$

These equivalence classes can be considered as single points in a new space called Y.

(a) Draw a picture of X showing some typical equivalence classes.
(b) Can you describe Y as a simple shape or otherwise geometrically? Remember that in the space Y, points that were equivalent in X are "glued together."

b) Let D be the "filled-in disk" $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. We can assign an equivalence relation to D so that two points (x_1, y_1) and (x_2, y_2) in D are equivalent if:
- they are the same point, or
- $x_1^2 + y_1^2 = 1, x_2^2 + y_2^2 = 1, x_1 = x_2$ and $y_1 = -y_2$

These equivalence classes can be considered as single points in a new space called B.

(a) Draw a picture of D showing some typical equivalence classes.
(b) Can you describe B as a simple shape or otherwise geometrically?

The spaces Y and B are called *quotient spaces.*

2 Let A be a set and P be a collection of subsets of A (that is, $P \subseteq \mathcal{P}(A)$). We say P is a **partition** of A if:

i) $\varnothing \notin P$.

ii) For all $b \in A$, there exists $B \in P$ such that $b \in B$.

iii) For any B and C in P, if $B \neq C$ then B and C are disjoint.

Prove the following:

a) Let Q be a set and let R be an equivalence relation on Q. Prove that the set of equivalence classes of R form a partition of Q.

b) Challenge: Suppose P is a partition of Q. Prove that there exists a unique equivalence relation R on Q such that $P = {}^{Q}/_{R}$

3 Let $\omega = \mathbb{N} \cup \{0\}$. In this problem, we will construct the integers from only the set ω.

a) Let's start by thinking about an ordered pair (a, b) from ω as representing the number $a - b$. Then, we'd like two ordered pairs (a, b) and (c, d) to be equal if they represent the same number, that is, $a - b = c - d$. However, ω is not closed under subtraction, so we will write this relation in terms of addition. Define the relation $\sim$ on $\omega \times \omega$ by:

$$(a, b) \sim (c, d) \text{ if and only if } a + c = b + d.$$

Prove that $\sim$ is an equivalence relation.

b) We would like to show that ${}^{\omega\times\omega}/_{\sim}$ behaves like $\mathbb{Z}$. We can now think about $[(a, b)]$ as representing the number $a-b$. Then, we'd like $[(a, b)]+[(c, d)]$ to represent $(a-b)+(c-d)$. We would like to define it without subtraction, so we'll write this as $(a+c)-(b+d)$, which is $[(a + c), (b + d)]$. Therefore, we will define addition as follows:

$$[(a, b)] + [(c, d)] = [(a + c, b + d)].$$

We will prove that our new addition is "well-defined," that is, that if we use different representatives for our equivalence classes, the addition still holds. Suppose $(e, f) \in [(a, b)]$ and $(g, h) \in [(c, d)]$. Using the definition above, prove that $[(a, b)] + [(c, d)] = [(e, f)] + [(g, h)]$.

c) Using part b) as a template, define multiplication on ${}^{\omega\times\omega}/_{\sim}$ and show that it is well-defined.

d) Challenge: define an equivalence relation R on $\mathbb{Z} \times \mathbb{N}$ so that ${}^{\mathbb{Z}\times\mathbb{N}}/_{R}$ behaves like the rational numbers.