ProofSpace Problem Set

Sets

Indexed Sets

Discussed Problems

For this assignment, some of our questions have less "building blocks" than you might be used to. Whenever confronted with indexed sets, it's a good idea to write out a few of the sets to get an idea of what's going on, and then proceed to answer the question asked.

1 Let $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$. For each $i \in I$, let

$$B_i = \{n \in \mathbb{N} \mid n < \frac{1}{i}\}.$$

Find the following:

a) $\bigcup_{i \in I} B_i$

b) $\bigcap_{i \in I} B_i$

2 For all $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{R} \mid \frac{1}{n} \leq x \leq n\}$.

a) $\bigcup_{n \in \mathbb{N}} A_n$

b) $\bigcap_{n \in \mathbb{N}} A_n$

c) $\bigcup_{n \in \mathbb{N}} A_n^c$

d) $\bigcap_{n \in \mathbb{N}} A_n^c$

e) $(\bigcup_{n \in \mathbb{N}} A_n)^c$

f) $(\bigcap_{n \in \mathbb{N}} A_n)^c$

3 For all $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{R} \mid \frac{1}{n} < x < 1 + n\}$. Let $B = \{x \in \mathbb{R} \mid -1 \leq x\}$.

a) $B \cup \bigcup_{n \in \mathbb{N}} A_n$

b) $B \cap \bigcap_{n \in \mathbb{N}} A_n$

c) $\bigcup_{n \in \mathbb{N}} (A_n \cup B)$

d) $\bigcap_{n \in \mathbb{N}} (A_n \cap B)$

e) Notice that some of your answers in (a)-(d) are the same. What property does this illustrate?

Evaluated Problems

1 Let $I = \{x \in \mathbb{R} \mid 0 < x\}$. For each $i \in I$, let

$$D_i = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 0 < x < i \text{ and } 0 < y < \frac{1}{i}\}.$$

Find the following. If necessary, you may describe your answers geometrically.

a) $\bigcup_{i \in I} D_i$

b) $\bigcap_{i \in I} D_i$

2 Let I be an indexed set and A_i be a collection of sets with $i \in I$. Prove DeMorgan's Laws for indexed sets, that is:

a) $\bigcup_{i \in I} A_i^c = (\bigcap_{i \in I} A_i)^c$

b) $\bigcap_{i \in I} A_i^c = (\bigcup_{i \in I} A_i)^c$

3 Let I be an indexed set and A_i be a collection of sets with $i \in I$. Let B be any set. Prove the distribution laws for indexed sets, that is:

a) $B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (A_i \cup B)$

b) $B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (A_i \cap B)$

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 5.5: 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13

Advanced Problems

1 In a previous problem (Sets/Operations/Advanced/2), we discussed the set-theoretic definition of an ordered pair. In this problem, we will develop the theory of ordered tuples further. An **ordered n-tuple** will be an ordered list $(x_1, x_2, ...x_n)$ of n elements, where $(x_1, x_2, ...x_n) = (y_1, y_2, ...y_n)$ if and only if $x_i = y_i$ for all natural numbers $i \leq n$.

Let A be a set. For $n \in \mathbb{N}$, we defined the set A^n to be the set $A^{n-1} \times A$. Thus, if $A = \mathbb{Z}$, an element of A^4 would might be $(((2, -6), 15), -1)$. We can now instead say elements of A^n are ordered n-tuples, so we might instead consider $(2, -6, 15, -1)$.

a) Let A be a set with m elements. Make a conjecture for the size of A^n. Prove your conjecture.

b) Suppose $I = \{1, 2, 3\}$. Then, we could write $\times_{i \in I} A$ to represent A^3. Virtually nobody uses this notation, however. Instead, we use the symbol $\prod$. (why might we use the letter "pi"?) In this example, we might use $\left(\prod_{i=1}^{3} A\right)$ to represent A^3. Suppose for all $i \in I$, $A_i = \mathbb{Z}$. Can you name an example of an element $\prod_{i \in \mathbb{N}} A_i$?

c) Can you name an example of an element of $\prod_{i \in \mathbb{R}} A_i$? Why or why not?

2 Recall that $A \cap B \subseteq A \cup B$. Use **just** the operations of union and power set to find a set D in terms of A and B such that $A \times B \subseteq D$. Hint: Use the set-theoretic definition of an ordered pair.