ProofSpace Problem Set

Preliminaries

Number Systems, Quantifiers, and Their Negations

Discussed Problems

1 Decide if the following statements are true or false. If a statement is false, provide a counterexample.

a) The real numbers are closed under addition.

b) The real numbers are closed under subtraction.

c) Multiplication is closed in the natural numbers.

d) The positive real numbers are closed under multiplication.

e) The real numbers are closed under division.

f) 2+5 is closed under addition in the integers.

g) The non-zero real numbers are closed under addition.

h) The negative real numbers are closed under subtraction.

i) The positive rational numbers are closed under multiplication.

j) The non-zero real numbers are closed under division.

2 Decide if the following statements are true or false. If false, provide a counterexample. If true, attempt a proof.

a) The rational numbers are closed under subtraction.

b) The non-zero integers are closed under division.

3 Recall the following definition from calculus:

A function f is **continuous** at a real number a means that for every ε that is greater than 0, there exists a δ that is greater than 0 such that if $|x-a| < \delta$, then $|f(x)-f(a)| < \varepsilon$.

Note that ε is the symbol "epsilon" and δ is the symbol "delta."

a) Write out the definition of "continuous at a" in symbols.

b) Write out the definition of "not continuous at a" in symbols.

4 Decide if the following sentences are true or false. Explain your conclusion.

a) $(\forall x \in \mathbb{Z})(x^2 \in \mathbb{N})$.

b) $(\exists y \in \mathbb{R})(y + y < y)$.

c) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x + y > y)$.

d) $(\exists y \in \mathbb{N})(\forall x \in \mathbb{N})(x + y > y)$.
Does this have the same meaning as the previous statement? Explain.

5 Negate the following propositions.

a) $(\forall x \in \mathbb{Z})(x + 5 \text{ is odd})$.

b) $(\exists y \in \mathbb{Q})(y^2 < y)$.

c) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{R})(xy = y)$.

Evaluated Problems

1 Decide if the following statements are true or false. If a statement is false, provide a counterexample.

a) The negative integers are closed under addition.

b) The negative integers are closed under subtraction.

c) The positive real numbers are closed under multiplication.

2 Are the rational numbers closed under multiplication? If not, provide a counterexample. If so, attempt a proof.

3 Consider the following definitions from algebra:

An operation $\star$ is **commutative** in a number system N means that for all x and y in N, $(x \star y) = (y \star x)$

A number system N with an operation $\star$ **has an identity** means that there is a number z in N such that for every y in N, $y \star z = y$ and $z \star y = y$.

a) Write out in symbols what it means for $\star$ to be commutative in N.

b) Write out in symbols what it means for a number system N with an operation $\star$ to have an identity.

c) Write out in symbols what it means for $\star$ to **not be** commutative in N.

d) Write out in symbols what it means for a number system N with an operation $\star$ to **not** have an identity.

4 Decide if the following sentences are true or false. Explain your conclusion.

a) $(\forall x \in \mathbb{Q})(x^2 \in \mathbb{Q})$.

b) $(\exists y \in \mathbb{R})(y + y = y)$.

c) $(\exists y \in \mathbb{R})(\forall p \in \mathbb{N})(p > y)$

d) $(\forall y \in \mathbb{R})(\exists p \in \mathbb{N})(p > y)$
Does this have the same meaning as the previous statement? Explain.

e) $(\exists p \in \mathbb{N})(\forall y \in \mathbb{R})(p > y)$
Does this have the same meaning as the previous statement? Explain.

5 Negate the following propositions.

a) $(\forall x \in \mathbb{Z})(x$ is a multiple of 3$)$.

b) $(\exists n \in \mathbb{N})(n^3 - n > n^2)$.

c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{Z})(\frac{x}{y} > x)$.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 1.1: 9
Sec. 2.4: 1, 2, 3, 4, 5, 10

Advanced Problems

1 The following puzzle is adapted from Raymond Smullyan's book, *What Is the Name of This Book?*. Suppose the following statements are true:

i) Everyone is afraid of Dracula.

ii) Dracula is afraid only of your professor.

Derive a conclusion.

2 The following puzzle is adapted from Raymond Smullyan's book, *What Is the Name of This Book?*. Consider the following proof that a unicorn exists.

> "I wish to prove to you that there exists a unicorn. To do this, it obviously suffices to prove the (possibly) stronger statement that there exists an existing unicorn. (By an existing unicorn I of course mean a unicorn which exists.) Surely if there exists an existing unicorn, then there must exist a unicorn. So all I have to do is prove that an existing unicorn exists. Well, there are exactly two possibilities:
>
> (1) An existing unicorn exists.
>
> (2) An existing unicorn does not exist.
>
> Possibility (2) is clearly contradictory: How could an existing unicorn not exist? Just as it is true that a blue unicorn is necessarily blue, an existing unicorn must necessarily be existing."

Let us all get together on this one and agree that unicorns do not exist (sorry, unicorn believers.) Then this proof obviously has a gaping flaw. The flaw lies in the interpretation of the word "an," which sometimes means "every" and other times means "at least one." For example, if we say, "An ostrich is a bird," we mean that ALL ostriches are birds. On the other hand, if we say "An ostrich is attacking me," we hopefully do not mean that all ostriches are attacking us, but that there exists an ostrich that is attacking us. So when we say, "An existing unicorn exists," it is unclear whether we are saying that ALL existing unicorns exist or that there exists an existing unicorn. Provide an explanation, using the idea of negation, to explain why the proof is invalid regardless of the interpretation.

3 When we proved the rational numbers were closed under addition, we used the fact that, because $b \neq 0$ and $d \neq 0$, $bd \neq 0$. In this problem, we will consider some nuances of this property.

a) What is the contrapositive of "$(b \neq 0)$ and $(d \neq 0) \implies (bd \neq 0)$?"

b) A number system is said to *have no zero divisors* if the statement from part a) is true. For example, $\mathbb{Z}$ has no zero divisors. Explain how we might use this fact to show that if $x^2 - 3x + 2$, then $x = 2$ or $x = 1$.

c) (This problem is easiest if you have had a course in linear algebra or multivariable calculus. If you have not, try to construct a *finite* number system.) Can you think of a mathematical system that *does* have zero divisors? That is, can you think of a system M such that $b \neq 0$ and $d \neq 0$, but $b \times d = 0$?