ProofSpace Problem Set

Proof Techniques

Direct Proofs and Counterexamples

Discussed Problems

For each of the following, prove or disprove. If a biconditional is false, prove or disprove each direction.

1 Let a, b, and c be integers.

a) If $a \mid b$ and $b \mid c$, then $a \mid c$.

b) If $a \mid bc$, then $a \mid b$ or $a \mid c$.

c) If $ab \mid c$, then $a \mid c$ and $b \mid c$.

d) If $a \mid b$ and $a \mid c$, then $a \mid b + c$.

2 Let x be an integer.

a) If x is odd, then $x^2 + 2x + 7$ is even.

b) If $x^2 + x$ is even, then x is even.

c) If x is odd, then $8 \mid 4x^2 + 12$.

d) If $3x + 1$ is odd, then $9x^2 + 6x + 1$ is even.

3 Let x and y be real numbers.

a) x is rational if and only if xy is rational.

b) If x is rational, then x^2 is rational.

4 Let n be a natural number. If a is an integer, then $a \equiv a \pmod{n}$.

This is called the **reflexive property**. Note that we have already seen that congruence modulo n satisfies the **symmetric property**:

$$\text{For all integers } a \text{ and } b, \text{ if } a \equiv b \pmod{n}, \text{ then } b \equiv a \pmod{n}.$$

Evaluated Problems

For each of the following, prove or disprove. If a biconditional is false, prove or disprove each direction.

1 Let a, b, and c be integers.

a) If $a \mid b + c$, then $a \mid b$ or $a \mid c$.

b) If $a \mid b$ and $a \mid c$, then $a \mid xb + yc$ for any integers x and y.

c) Suppose a is positive. If $a \mid b$ and $a \mid b + 1$, then $a = 1$.

2 Let x be an integer.

a) If x is odd, then $x^2 + 2x + 1$ is even.

b) If $x + 4$ is odd, then $x^2 + 7x + 12$ is even.

3 Let x and y be real numbers. Then, x is rational if and only if $x + y$ is rational.

4 Let n be a natural number. Then congruence modulo n satisfies the **transitive property**:

For all integers a, b, and c, if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

Supplemental Problems

Mathematical Reasoning: Writing and Proof, Online Version 2.0, by Ted Sundstrom:
Sec. 3.1: 1, 2, 3, 5, 7, 9, 10, 11, 12, 19

Advanced Problems

An integer p is **prime** if $(a \mid p) \Rightarrow (a = \pm 1) \vee (a = \pm p)$.

Let a be an integer. If $3 \mid a^2$, then $3 \mid a$. (We will soon have the tools to prove this.)

1 Find all primes p such that $p + 1$ is a perfect cube.

2 The Number Theoretic Troll (NTT) gives you a number whose digits are 3000 0's and 3000 1's in some order.

a) The NTT says the number is prime. Is he telling the truth or lying?

b) The NTT says the number is a perfect square. Is he telling the truth or lying?