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Necessary conditions for controllability of nonlinear networked control systems

Cesar O. Aguilar[1] and Bahman Gharesifard[2]

***Abstract*— In this paper, we study the controllability of nonlinear networked systems. In particular, we describe how graph symmetries combined with dynamic symmetries result in a loss of controllability in nonlinear leader-follower networks. Our result generalizes those of Rahmani et al. (2009) who considered the case of a linear consensus-type dynamics, namely the unweighted Laplacian network flow. We consider several nonlinear network control systems that have been previously studied in the literature and characterize the presence of leader-follower graph symmetries that result in the lack of controllability.**

I. INTRODUCTION

Many modern engineering and economic systems consist of collections of smaller subsystems, interconnected to each other over a network. Examples include distributed energy resources, oscillator synchronization, and distributed robotic networks, see [1], [2], [3] and also cascades of information and opinions in social networks [4]. Controllability and stabilizability of networked control systems are fundamental problems in distributed control theory. For linear systems, it is by now well-understood that the topology of the network, alongside with limitation of control inputs, influences the possibility of controlling the system. It is, for example, known that certain symmetries in the placement of control vertices can cause lack of controllability in consensus dynamics [5]. On the stabilizability front, the recent work [6] investigates the limitations imposed by the topology of interconnections in linear networked control systems via the notion of sparse stable matrices. Other examples include [7] in the context of passivity-based design.

The focus of this current work is on the controllability of networked control systems. In contrast to [5], a particular interest of this paper will be on identifying easy-to-verify obstructions to controllability of networked control systems whose evolutions are driven by nonlinear dynamics, not necessarily of the consensus type. Examples of such systems include the absolute, relative, and disagreement nonlinear flows presented in [8] and virus spread in networks [9].

Statement of Contributions: The contributions of this paper are threefold. We first introduce the notion of symmetries for network flows whose evolutions are not necessarily linear. In contrast to similar notions for linear flows, symmetries of both 1) the underlying network and 2) the dynamical system play important roles in this definition. We characterize these symmetries for various examples, including the absolute, relative, and disagreement nonlinear flows and also the virus spreading dynamics. We then introduce the notion of leader-follower network flow. As our second contribution, we give a necessary condition for controllability of general networked control systems with real-analytic flows. Among other things, this result demonstrates that the existing obstructions to controllability in the literature on multi-agent systems are instances of lack of accessibility. As our last contribution, we present a suit of examples, including the loss of controllability in spite of dynamic asymmetry and loss of controllability in spite of leader symmetry, demonstrating that these results are far from being sufficient, even when lack of accessibility is concerned. We suggest some future directions for improving these results in our conclusion section.

Organization: Section II contains preliminaries on graph theory and nonlinear controllability. In Section III, we introduce and study the symmetries of a network dynamical system in terms of the automorphisms of its underlying graph. Section IV contains our results on controllability of nonlinear networked control systems. Finally, Section V gathers our conclusions and ideas for future work.

II. MATHEMATICAL PRELIMINARIES

Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the real and nonnegative real numbers. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field on $\mathbb{R}^n$ and let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping. We say that F is ϕ-invariant if $(D\phi(x))F(x) = F(\phi(x))$ for all $x \in \mathbb{R}^n$, where $D\phi(x)$ denotes the Jacobian matrix of ϕ at x. The Lie bracket of two vector fields f and g will be denoted by $[f, g]$. Given a mapping $\phi : X \to X$ we let $\mathrm{Fix}(\phi) = \{x \in X \mid \phi(x) = x\}$ denote its fixed point set.

A. Graph theory

We introduce some basic notions from graph theory [10]. A *graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a *vertex* set $\mathcal{V}$ and an *edge* set $\mathcal{E} \subseteq [\mathcal{V}]^2 := \{\{v, w\} \mid v, w \in \mathcal{V}\}$. The *order* of the graph $\mathcal{G}$ is the cardinality of its vertex set $\mathcal{V}$. In this paper we deal only with graphs of finite order. The *neighbors* of $v \in \mathcal{V}$ is the set $\mathcal{N}_v := \{w \in \mathcal{V} \mid \{v, w\} \in \mathcal{E}\}$ and the *degree* of v is $d_v := |\mathcal{N}_v|$. A *path* in $\mathcal{G}$ is a sequence of edges of the form $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}$. The graph $\mathcal{G}$ is *connected* if there is a path between any pair of vertices. Henceforth, when not explicitly stated, we fix an ordering on the vertex set $\mathcal{V}$ so that we may write $\mathcal{V} = \{v_1, \ldots, v_n\}$, where n is the order of $\mathcal{G}$. The *adjacency matrix* of $\mathcal{G}$ is the $n \times n$ matrix A defined as $\mathsf{A}_{i,j} = 1$ if $\{v_i, v_j\} \in \mathcal{E}$ and $\mathsf{A}_{i,j} = 0$ otherwise.

[1]Cesar O. Aguilar is with the Mathematics Department, California State University, Bakersfield, CA 93311, USA caguilar24@csub.edu
[2]Bahman Gharesifard is with the Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada bahman@mast.queensu.ca

A mapping $\varphi : \mathcal{V} \to \mathcal{V}$ is an *automorphism* of $\mathcal{G}$ if φ is a bijection and $\{v, w\} \in \mathcal{E}$ implies that $\{\varphi(v), \varphi(w)\} \in \mathcal{E}$. An automorphism φ of $\mathcal{G}$ induces a linear transformation on $\mathbb{R}^n$ whose matrix representation in the standard basis is a permutation matrix, i.e., as a linear mapping φ acts as a permutation on the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. When no confusion arises, we will denote this linear transformation with the same symbol φ.

B. Nonlinear controllability

We review some basic notions from nonlinear controllability theory. Consider the control system

$$\dot{x} = f(x, u), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth mapping. The control functions $u : [0, T] \to \mathbb{R}^m$ for (1) will be assumed to be piecewise constant. The accessible set of (1) from a point $x_0 \in \mathbb{R}^n$ at time $T \geq 0$ will be denoted by $\mathcal{A}(x_0, T)$ and consists of all end-points $\gamma(T)$, where $\gamma : [0, T] \to \mathbb{R}^n$ is a controlled trajectory of (1), i.e. $\dot{\gamma} = f(\gamma(t), u(t))$ for some control $u : [0, T] \to \mathbb{R}^m$. The accessible set of (1) from x_0 up to time $T \geq 0$ is the union $\mathcal{A}(x_0, \leq T) := \cup_{0 \leq \tau \leq T} \mathcal{A}(x_0, \tau)$.

Definition 2.1: We say that (1) is *accessible* from x_0 if for every $T > 0$ the set $\mathcal{A}(x_0, \leq T)$ contains a non-empty interior.

Remark 2.1: If x_0 is a controlled equilibrium of (1), i.e., $f(x_0, u_0) = 0$ for some $u_0 \in \mathbb{R}^m$ then $\mathcal{A}(x_0, \leq T) = \mathcal{A}(x_0, T)$ for every $T \geq 0$. Indeed, any $x \in \mathcal{A}(x_0, \tau)$ where $\tau < T$ can be reached from x_0 in time T by concatenating the constant control $u(t) = u_0$ on the interval $[0, T - \tau)$ with a control that steers x_0 to x in time τ.

Let $\mathcal{F}$ be the family of vector fields on $\mathbb{R}^n$ defined by

$$\mathcal{F} := \{f(\cdot, u) \mid u \in \mathbb{R}^m\}.$$

We denote by $\mathrm{Lie}(\mathcal{F})$ the Lie algebra of vector fields generated by $\mathcal{F}$, i.e., the smallest Lie subalgebra of vector fields containing $\mathcal{F}$. The subspace generated by evaluating the vector fields in $\mathrm{Lie}(\mathcal{F})$ at the point $x \in \mathbb{R}^n$ will be denoted by $\mathrm{Lie}_x(\mathcal{F})$. We say that $\mathcal{F}$ satisfies the *Lie algebra rank condition* (LARC) at x if $\mathrm{Lie}_x(\mathcal{F})$ has full dimension n. A proof of the following fundamental result can be found in [11].

Theorem 2.1 ([11]): Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a real-analytic mapping. The control system (1) is accessible from x_0 if and only if $\mathcal{F} = \{f(\cdot, u) \mid u \in \mathbb{R}^m\}$ satisfies the LARC at x_0.

III. Symmetries of Network Flows

In this section, we introduce network dynamic flows and a type of invariance (or symmetry) that respects both the underlying network and dynamics. We give examples of some recently studied network flows and characterize their invariance. We also introduce leader-follower network flows which are control systems obtained by choosing leader nodes in a network to act as control inputs.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set $\mathcal{V} = \{1, 2, \dots, n\}$. Let $\dot{x} = F(x)$ be a dynamical system on $\mathbb{R}^n$ with component functions F_i for $i \in \{1, 2, \dots, n\}$.

Definition 3.1: **(Dynamical systems on networks):** We say that F is a *dynamical system on* $\mathcal{G}$, or a *network flow on* $\mathcal{G}$, if $F_i(x_1, \dots, x_n)$ is independent of x_j for $j \in \mathcal{V} \backslash (\mathcal{N}_i \cup \{i\})$, for all $i \in \{1, 2, \dots, n\}$. In other words, F is a network flow on $\mathcal{G}$ if the dynamics of each node depends possibly only on its state and the states of its neighbors.

Definition 3.2: **(Symmetries of network flows):** Let F be a network flow on $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We say that $\varphi : \mathcal{V} \to \mathcal{V}$ is a *symmetry* of the network flow F if φ is an automorphism of $\mathcal{G}$ and F is φ-invariant.

Let $\varphi : \mathcal{V} \to \mathcal{V}$ be an automorphism of $\mathcal{G}$. Identifying φ as a linear mapping on $\mathbb{R}^n$, it is not hard to see that a network flow F on $\mathcal{G}$ *is* φ-invariant if and only if

$$F_{\varphi(i)}(x) = F_i(\varphi(x)), \tag{2}$$

for all $x \in \mathbb{R}^n$.

A. Examples of Network Flows

In this section, we define several general network flows and characterize the presence of symmetries.

Example 3.1: **(φ-invariant flows on graphs):** Consider the graph $\mathcal{G}$ depicted in Figure 1.

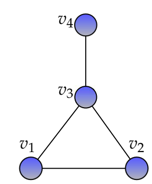

Fig. 1. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\}$ having automorphism $\varphi : \mathcal{V} \to \mathcal{V}$ defined by $\varphi(v_1) = v_2$ and $\varphi(v_2) = v_1$ and fixing v_3 and v_4.

A network flow $F : \mathbb{R}^4 \to \mathbb{R}^4$ on $\mathcal{G}$ takes the form

$$F(x) = \begin{pmatrix} a(x_1, x_2, x_3) \\ b(x_1, x_2, x_3) \\ c(x_1, x_2, c_3, x_4) \\ d(x_3, x_4) \end{pmatrix}$$

for smooth functions a, b, c, d. Let φ be the automorphism of $\mathcal{G}$ defined by $\varphi(v_1) = v_2$, $\varphi(v_2) = v_1$, and fixing v_3 and v_4. Then F is φ-invariant if and only if

$$b(x_2, x_1, x_3) = a(x_1, x_2, x_3)$$

and

$$c(x_1, x_2, x_3, x_4) = c(x_2, x_1, x_3, x_4).$$

Example 3.2: **(φ-invariance for the Laplacian flow):** A *weighted graph* is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a mapping $\mathrm{wgt} : \mathcal{E} \to \mathbb{R}_{>0}$. The adjacency matrix A of $\mathcal{G}$ is defined as $\mathsf{A}_{i,j} = \mathrm{wgt}(\{i, j\})$ if $\{i, j\} \in \mathcal{E}$ and zero otherwise. The *Laplacian flow* on $\mathcal{G}$ is the dynamics $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(x) = -\mathsf{L}x$ where $\mathsf{L} = \mathrm{diag}(d_1, \dots, d_n) - \mathsf{A}$. Let φ be an automorphism of $\mathcal{G}$. Using the fact that $d_{\varphi(i)} = d_i$

(φ preserves degree), it is straightforward to verify using (2) that the Laplacian flow is φ-invariant if and only if

$$\sum_{\ell \in \mathcal{N}_{\varphi(i)}} \mathsf{A}_{\varphi(i),\ell} x_\ell = \sum_{j \in \mathcal{N}_i} \mathsf{A}_{i,j} x_{\varphi(j)}, \\ = \sum_{\ell \in \mathcal{N}_{\varphi(i)}} \mathsf{A}_{i,\varphi^{-1}(\ell)} x_\ell.$$

Therefore, F is φ-invariant if and only if $\mathsf{A}_{\varphi(i),\ell} = \mathsf{A}_{i,\varphi^{-1}(\ell)}$ for $\ell \in \mathcal{N}_{\varphi(i)}$, or equivalently that $\mathsf{A}_{i,j} = \mathsf{A}_{\varphi(i),\varphi(j)}$ for every $\{i,j\} \in \mathcal{E}$. Clearly, if all the weights $\mathsf{A}_{i,j}$ are the same, an automorphism of the graph $\mathcal{G}$ is automatically an invariance of the Laplacian dynamics $F(x) = -\mathsf{L}x$. This is the setting considered in [5].

Example 3.3: **(Absolute, relative, and disagreement nonlinear flows):** In our definitions below, $f_i : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f_i(0) = 0$ for $i \in \{1, 2, \ldots, n\}$. Following [8], below we present some network flows:

- We say that F is an *absolute nonlinear flow* on $\mathcal{G}$ if

$$F_i(x) = \sum_{j \in \mathcal{N}_i} f_i(x_i) - f_j(x_j). \tag{3}$$

  In other words, each node transmits a value which is a function of its state only.
- We say that F is a *relative nonlinear flow* on $\mathcal{G}$ if

$$F_i(x) = \sum_{j \in \mathcal{N}_i} f_i(x_i - x_j). \tag{4}$$

  In other words, each node transmits a value that depends only on the difference between its state and the states of its neighboring nodes.
- We say that F is a *disagreement nonlinear flow* on $\mathcal{G}$ if

$$F_i(x) = f_i\left(\sum_{j \in \mathcal{N}_i} (x_i - x_j)\right). \tag{5}$$

  In other words, each node transmits a value that is a function of the corresponding component in the disagreement vector.

Lemma 3.1: **(Characterization of φ-invariance for absolute, relative, and disagreement nonlinear flows):** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be any of the network flows (3), (4), or (5). Let φ be an automorphism of $\mathcal{G}$. Then F is φ-invariant if and only if $f_i \equiv f_{\varphi(i)}$ for all $i \in \mathcal{V}$.

Proof: Fix $i \in \mathcal{V}$ and let $k = \varphi(i)$. First of all, it is clear that

$$(D\varphi(x)F(x))_i = F_{\varphi(i)}(x) = F_k(x)$$

irrespective of the flow F. Suppose that F is the absolute flow (3). Then,

$$F_i(\varphi(x)) = \sum_{j \in \mathcal{N}_i} f_i(x_{\varphi(i)}) - f_j(x_{\varphi(j)}), \\ = \sum_{j \in \mathcal{N}_i} f_i(x_k) - f_j(x_{\varphi(j)}), \\ = \sum_{\ell \in \mathcal{N}_k} f_i(x_k) - f_{\varphi^{-1}(\ell)}(x_\ell),$$

where the second equality follows from $\varphi(\mathcal{N}_i) = \mathcal{N}_k$. If $f_j \equiv f_{\varphi(j)}$ for all $j \in \mathcal{V}$ then

$$F_i(\varphi(x)) = \sum_{\ell \in \mathcal{N}_k} f_k(x_k) - f_\ell(x_\ell) = F_k(x),$$

and thus F is φ-invariant. On the other hand if F is φ-invariant we have $F_k(x) = F_i(\varphi(x))$ for all x, that is,

$$\sum_{\ell \in \mathcal{N}_k} f_k(x_k) - f_\ell(x_\ell) = \sum_{\ell \in \mathcal{N}_k} f_i(x_k) - f_{\varphi^{-1}(\ell)}(x_\ell)$$

for all x. Setting $x_\ell = 0$ for all $\ell \in \mathcal{N}_k$ we obtain that $|\mathcal{N}_k| f_k(x_k) = |\mathcal{N}_k| f_i(x_k)$. By connectivity of $\mathcal{G}$ it must hold that $|\mathcal{N}_k| \geq 1$ and therefore $f_k(x_k) = f_i(x_k)$ for all $x_k \in \mathbb{R}$, i.e., $f_i \equiv f_{\varphi(i)}$. This proves the claim for the case that F is the absolute flow.

If F is the relative flow then

$$F_i(\varphi(x)) = \sum_{j \in \mathcal{N}_i} f_i(x_{\varphi(i)} - x_{\varphi(j)}), \\ = \sum_{j \in \mathcal{N}_i} f_i(x_k - x_{\varphi(j)}), \\ = \sum_{\ell \in \mathcal{N}_k} f_i(x_k - x_\ell).$$

Therefore, if $f_i \equiv f_k$ (i.e., $f_i \equiv f_{\varphi(i)}$) then $F_i(\varphi(x)) = F_k(x)$ and this proves that F is φ-invariant. Now suppose that F is φ-invariant. As in our previous computations, we have

$$\sum_{\ell \in \mathcal{N}_k} f_k(x_k - x_\ell) = \sum_{\ell \in \mathcal{N}_k} f_i(x_k - x_\ell),$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. In particular, if we set $x_\ell = x_k - s$ for all $\ell \in \mathcal{N}_k$ for arbitrary $s \in \mathbb{R}$ then $|\mathcal{N}_k| f_k(s) = |\mathcal{N}_k| f_i(s)$. By the connectivity of $\mathcal{G}$ we must have $|\mathcal{N}_k| \geq 1$, from which we can conclude that $f_k(s) = f_i(s)$ for all $s \in \mathbb{R}$, i.e., $f_i \equiv f_{\varphi(i)}$. This proves the claim when F is the relative flow.

The proof of the disagreement flow is similar and we leave it to the reader. ■

Example 3.4: **(Virus spreading dynamics):** As a last example, we present a general form of the virus spreading dynamics considered in [9]. The dynamics $F = (F_1, \ldots, F_n)$ is given by

$$F_i(x) = -\delta_i x_i + (1 - x_i) \sum_{j \in \mathcal{N}_i} f_j(x_j), \tag{6}$$

where $\delta_i > 0$.

Using similar calculations as in the proof of Lemma 3.1 one can prove the following.

Lemma 3.2: **(Characterization of φ-invariance for virus spreading dynamics):** Let F be the network flow (6) defined on $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and let φ be an automorphism of $\mathcal{G}$. Then F is φ-invariant if and only if $\delta_i = \delta_{\varphi(i)}$ and $f_i \equiv f_{\varphi(i)}$ for all $i \in \mathcal{V}$.

One way to turn a network flow into a control system is to specify nodes whose states can be controlled directly and consider the control system whose state are states of the remaining nodes. We elaborate on this idea in the following section.

B. Leader-Follower Network Flows

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a network flow on $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$. Partition the vertex set $\mathcal{V}$ into an ordered set of *leaders* $\mathcal{V}_\ell = \{i_1, \ldots, i_M\}$ and an ordered set of *followers* $\mathcal{V}_f = \{j_1, \ldots, j_N\}$, i.e., $1 \leq i_1 < \cdots < i_M \leq n$ and $1 \leq j_1 < \cdots < j_N \leq n$, $\mathcal{V}_\ell \cap \mathcal{V}_f = \emptyset$ and $\mathcal{V} = \mathcal{V}_\ell \cup \mathcal{V}_f$. Let $z = (x_{j_1}, \ldots, x_{j_N})$ denote the follower states and let $u = (x_{i_1}, \ldots, x_{i_M})$ denote the leader states. It is clear that there is a $n \times n$ permutation matrix P such that $(z, u) = Px$. Let $g = (g_1, \ldots, g_N) : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ be defined by

$$g_k(z, u) = F_{j_k}(P^{-1}(z, u)),$$

for $k \in \ldots \{1, \ldots, N\}$. The control system

$$\dot{z} = g(z, u)$$

is called the *leader-follower network flow on $\mathcal{G}$ induced by the leaders $\mathcal{V}_\ell$*. The graph $\mathcal{G}$ is said to be *leader symmetric with respect to the leaders $\mathcal{V}_\ell$* if there exists a nonidentity automorphism φ of $\mathcal{G}$ such that $\mathcal{V}_\ell \subset \text{Fix}(\varphi)$. In this case, it is not hard to see that $\varphi|_{\mathcal{V}_f}$ is a nonidentity automorphism of the subgraph $\mathcal{G}_f \subseteq \mathcal{G}$ induced by the followers $\mathcal{V}_f$.

IV. Necessary conditions for controllability of networked control systems

In this section, we present the main result of this paper (Theorem 4.1). Let us first introduce some notation. Let $\pi_{n-1} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto the first $(n-1)$-coordinates. Given a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ we let $\tilde{\varphi} : \mathbb{R}^{(n-1)} \to \mathbb{R}^{(n-1)}$ be the mapping defined by $\tilde{\varphi}(z) = \pi_{n-1}(\varphi(z, 0))$. Hence, when φ is linear, $\tilde{\varphi}$ is the linear mapping restricted to $\text{span}\{e_1, \ldots, e_{n-1}\}$.

Proposition 4.1: **(A necessary condition for controllability):** Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that acts as a permutation on the standard basis $\{e_1, \ldots, e_n\}$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field and suppose that F is φ-invariant. Suppose that $\text{Fix}(\varphi)$ is non-empty and assume without loss of generality that $e_n \in \text{Fix}(\varphi)$. Let $z_i = x_i$, let $g_i(z, u) := F_i(z_1, \ldots, z_{n-1}, u)$ for $i \in \{1, 2, \ldots, n-1\}$, and let $g = (g_1, \ldots, g_{n-1})$. Then all of the trajectories of the control system

$$\dot{z} = g(z, u) \tag{7}$$

initiating from $z = 0$ lie in the subspace $\text{Fix}(\tilde{\varphi}) \subset \mathbb{R}^{n-1}$. In particular, if φ is a non-identity transformation then (7) is not accessible from $z = 0$.

Proof: Let $(z, u) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then

$$\begin{aligned} g(\tilde{\varphi}(z), u) &= \pi_{n-1}(F(\varphi(z, u))), \\ &= \pi_{n-1}(\varphi F(z, u)), \\ &= \tilde{\varphi}(\pi_{n-1}(F(z, u))), \\ &= \tilde{\varphi}(g(z, u)), \end{aligned} \tag{8}$$

where the second equality follows from φ-invariant of F and the third equality follows from the fact that φ is a permutation matrix and fixes e_n. Let $z \in \text{Fix}(\tilde{\varphi})$ and let $u \in \mathbb{R}$ be arbitrary. Clearly, $g(\tilde{\varphi}(z), u) = g(z, u)$ and therefore from (8) we have

$$g(z, u) = \tilde{\varphi}(g(z, u)),$$

that is, $g(z, u) \in \text{Fix}(\tilde{\varphi})$. It follows that any trajectory of (7) initiating from $z = 0 \in \text{Fix}(\tilde{\varphi})$ remains in the subspace $\text{Fix}(\tilde{\varphi})$, and this proves the first claim. Now, if φ is not the identity transformation on $\mathbb{R}^n$ then $\tilde{\varphi}$ cannot be the identity transformation on $\mathbb{R}^{n-1}$ because φ fixes e_n. It follows that $\text{Fix}(\tilde{\varphi})$ is a non-trivial subspace in $\mathbb{R}^{n-1}$. Thus, the accessible set of (7) from $z = 0$ does not contain an open set in $\mathbb{R}^{n-1}$ and this completes the proof. ■

The following theorem, whose proof follows immediately from Proposition 4.1, is the main result of this paper.

Theorem 4.1: **(Nontrivial φ-invariance leads to lack of network flow controllability):** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a network flow on $\mathcal{G}$. Let φ be a symmetry of the network flow F. If φ is not the identity mapping then for any vertex $j \in \text{Fix}(\varphi)$ chosen as the leader, the resulting leader-follower network flow on $\mathcal{G}$ induced by $\mathcal{V}_\ell = \{j\}$ is not accessible from the origin in $\mathbb{R}^{n-1}$.

The assumption that F be φ-invariant is necessary for the conclusion of Theorem 4.1 to remain true. In other words, contrary to the special case of the Laplacian flow [5] with all equal weights, one should not expect that a leader symmetry of the *graph* is sufficient for a lack of controllability. To see this consider the following example.

Example 4.1: **(Leader graph-symmetry is not sufficient for loss of controllability):** Consider the network $\mathcal{G}$ depicted in Figure 1. It is clear that $\mathcal{G}$ is leader symmetric with respect to $\mathcal{V}_\ell = \{v_4\}$ with automorphism $\varphi(v_1) = v_2$, $\varphi(v_2) = v_1$, and v_3 and v_4 fixed. Let F be the relative nonlinear flow (4) on $\mathcal{G}$ with all functions $f_i(z) = a_i z$ being linear. The resulting leader-follower network flow induced by $\mathcal{V}_\ell$ is

$$\begin{aligned} \dot{z}_1 &= a_1(z_1 - z_2) + a_1(z_1 - z_3), \\ \dot{z}_2 &= a_2(z_2 - z_1) + a_2(z_2 - z_3), \\ \dot{z}_3 &= a_3(z_3 - z_1) + a_3(z_3 - z_2) + a_3(z_3 - u). \end{aligned} \tag{9}$$

Let (A, B) denote the linear system corresponding to (9). One computes that

$$\det([B \ AB \ A^2B]) = 3a_1a_2a_3^3(a_1 - a_2). \tag{10}$$

Now, applying Lemma 3.1 to the relative nonlinear flow on $\mathcal{G}$, we deduce that $\mathcal{G}$ is φ-invariant if and only if

$a_1 = a_2$. On the other hand, from (10) we see that the leader-follower network (9) is controllable if and only if $a_1 \neq a_2$. Hence, although $\mathcal{G}$ is leader symmetric with respect to $\mathcal{V}_\ell = \{v_4\}$, the leader-follower dynamics induced by $\mathcal{V}_\ell$ may be controllable. We note that in this example, the lack of dynamic symmetry ($a_1 = a_2$) gives controllability (this may not be true for a nonlinear network flow, see Example 4.2).

As pointed out in [5], the presence of a leader dynamic-symmetry is also not a necessary condition for a lack of controllability. We now give an example of a nonlinear leader-follower network flow whose lack of controllability is independent of a dynamic symmetry.

Example 4.2: **(Loss of controllability in spite of dynamic asymmetry):** Consider again the network $\mathcal{G}$ depicted in Figure 1 and let as before $\mathcal{V}_\ell = \{v_4\}$ be the leader set and let φ the the automorphism permuting v_1 and v_2 and fixing v_3 and v_4. Let F be a network flow on $\mathcal{G}$ whose leader-follower network induced by $\mathcal{V}_\ell = \{v_4\}$ is

$$\begin{aligned} \dot{z}_1 &= (z_1 - z_2) + (z_1 - z_3), \\ \dot{z}_2 &= (z_2 - z_1) + (z_2 - z_3), \\ \dot{z}_3 &= (z_3 - z_1) + (z_3 - z_2) + c(z_1, z_2, z_3)u. \end{aligned} \tag{11}$$

From (2), it is not hard to see that irrespective of the component F_4 of F, the flow F is φ-invariant if and only if $c(z_1, z_2, z_3) = c(z_2, z_1, z_3)$. We claim that (11) is not accessible from the origin $z_0 = 0 \in \mathbb{R}^3$ for any choice of real-analytic function c. To prove our claim, we first note that (11) is a control-affine system. Let f and g denote the drift and control-input vector field associated to (11), respectively. As the vector fields f and g are real-analytic, and

$$\mathrm{Lie}(\{f + gu \mid u \in \mathbb{R}\}) = \mathrm{Lie}(\{f, g\}),$$

from Theorem 2.1 our claim will be proved if we can show that $\dim(\mathrm{Lie}_{z_0}(\{f, g\})) < 3$. To this end, consider the submanifold

$$S = \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1 - z_2 = 0\}$$

and we note that $z_0 \in S$. It is not hard to see that both f and g are tangent to S. By well-known facts about the Lie bracket [12], it follows that $[f, g]$ is also tangent to S. More generally, using the fact that $\mathrm{Lie}(\{f, g\})$ is spanned by Lie brackets of the form (see for instance [13])

$$[X_k, [X_{k-1}, [\cdots, [X_2, X_1]] \cdots]$$

where $X_i \in \{f, g\}$ and $k \geq 1$, it follows immediately that $\mathrm{Lie}_z(\{f, g\})$ is contained in the tangent space of S at z, and in particular at z_0. The manifold S is 2-dimensional and this proves the claim.

V. Conclusion

We present an easily verifiable necessary condition for accessibility of nonlinear networked control systems, without relying on computing the Lie algebra of vector fields. Our result sheds light into some obstruction to controllability of multi-agent systems imposed by symmetries, established previously in the literature for specific dynamics. Among other things, this result demonstrates that the interesting instances of lack of controllability, when the Lie algebra rank condition is satisfied, is outside what is characterized via this result and its other corollaries presented in the literature.

Future directions include characterizing other necessary conditions for accessibility of nonlinear networked control systems, studying controllability in directed settings, and identifying other checkable necessary conditions for controllability when the Lie algebra rank condition is satisfied.

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