Local controllability of control-affine systems with quadractic drift and constant control-input vector fields

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Abstract— **In this paper we study the small-time local controllability (STLC) property of polynomial control-affine systems whose drift vector field is a 2-homogeneous polynomial vector field and whose control-input vector fields are constant. Such systems arise in the study of controllability of mechanical control systems. Using control variations and rooted trees, we obtain a combinatorial expression for the Taylor series coefficients of a composition of flows of vector fields and use it to derive a high-order sufficient condition for STLC for these systems. The resulting condition is stated in terms of the image of the control-input subspace under the drift vector field and is therefore invariant under (linear) feedback transformations.**

I. Introduction

In this paper we consider the local controllability problem of polynomial control-affine systems

$$\Sigma : \dot{z} = X_0(z) + \sum_{i=1}^{m} u_i X_i(z) \tag{1}$$

where $X_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous polynomial vector field of integer degree $d \geq 1$, i.e., $X_0(\lambda z) = \lambda^d X_0(z)$ for all $\lambda \in \mathbb{R}$ and $z \in \mathbb{R}^n$, and $X_1, \ldots, X_m : \mathbb{R}^n \to \mathbb{R}^n$ are constant vector fields. The controls $u : [0,T] \to \mathbb{R}^m$ for Σ are assumed to be piecewise constant and take their values in a symmetric and compact subset $U \subset \mathbb{R}^m$ containing the origin in its interior. By symmetric we mean that $u \in U$ implies that $-u \in U$. We fix the initial condition of Σ to the origin $z_0 = 0 \in \mathbb{R}^n$. The reachable set of Σ from z_0 at time $T > 0$, denoted by $\mathcal{R}_\Sigma(z_0, T)$, is the set of all end-points $z(T)$ where $t \mapsto z(t)$ is a trajectory of Σ initiating from z_0 at time $t = 0$. We say that Σ is *small-time locally controllable* (STLC) from z_0 if $\mathcal{R}_\Sigma(z_0, T)$ contains z_0 in its interior for every $T > 0$.

Despite the very general sufficient conditions for STLC obtained in the landmark papers [15], [4], a complete understanding of the STLC property for (1) when d is even and $m \geq 2$ is an open problem. In this paper, we will focus on the quadratic case $d = 2$ and multi-input systems $m \geq 2$. The case when $d \in \{3, 5, 7, \ldots\}$ is odd is well understood [5] (see Section III for a summary) and is analogous to the the linear case $d = 1$. In the single-input case $m = 1$ and $d = 2$, it was shown in [13] that Σ is STLC from z_0 if and only if $n = 1$, that is, only when the state is a scalar. For the multi-input case and $d = 2$, in [13] a study is undertaken of the possibility of using linear input transformations so that for the new transformed system the first potential obstructions $[X_i, [X_i, X_0]](z_0)$, $i = 1, \ldots, m$, to STLC can be neutralized and an application of Sussmann's general theorem [15] would imply STLC. As shown in [13], the problem of neutralizing the potential obstructions $[X_i, [X_i, X_0]](z_0)$ is equivalent to the indefiniteness of the symmetric $\mathbb{R}$-bilinear map defined by the second derivative $\mathbf{D}^2 X_0(z_0)(\cdot, \cdot)$ of X_0. Indefiniteness of $\mathbf{D}^2 X_0(z_0)$ can be characterized in terms of the image of the quadratic map $\mathbb{R}^n \ni v \mapsto \mathbf{D}^2 X_0(z_0)(v, v)$ [6, pg. 413]. We make the following observation that, since for any $v \in \mathbb{R}^n$ and any vector field V such that $V(z_0) = v$ we have

$$[V, [V, X_0]](z_0) = \mathbf{D}^2 X_0(z_0)(v, v),$$

we can interpret the results of [13] as a statement about the image of the subspace $\operatorname{span}\{X_1(z_0), \ldots, X_m(z_0)\}$ under $\mathbf{D}^2 X_0(z_0)(\cdot, \cdot)$. This is analogous to the well-known fact that controllability for linear control systems can be fully characterized in terms of the image of the subspace spanned by the control-input vector fields under the 1-homogeneous drift vector field. The purpose of this paper is to fully exploit this analogy for systems of the form (1) with quadratic drift X_0 by studying the STLC property for the control-affine system

$$\Sigma_q : \begin{cases} \dot{x} = u \\ \dot{y} = q_1(x) + q_2(y). \end{cases} \tag{2}$$

In (2), the state variable is $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^r$, $u \in \mathbb{R}^m$ is the control variable, and $q_1 : \mathbb{R}^m \to \mathbb{R}^r$ and $q_2 : \mathbb{R}^r \to \mathbb{R}^r$ are homogeneous quadratic maps, i.e., $q_1(\lambda x) = \lambda^2 q_1(x)$ and $q_2(\lambda y) = \lambda^2 q_2(y)$ for all x, y, and $\lambda \in \mathbb{R}$. Our approach is to exploit the polynomial structure of (1) to explicitly compute certain high-order tangent vectors to the reachable set by studying the Taylor series coefficients of end-point mappings

$$(t_1, \ldots, t_p) \mapsto \Phi_{t_p}^{Z_p} \circ \Phi_{t_{p-1}}^{Z_{p-1}} \circ \cdots \circ \Phi_{t_1}^{Z_1}(z_0) \tag{3}$$

using *labeled rooted trees* [7]. In (3), $(t, z) \to \Phi_t^Z(z)$ denotes the flow of the vector field Z. Labeled rooted trees are used in [7] to give a combinatorial formula for the Taylor series coefficients of the solution of an ordinary differential equation. Following [7], we do the same for the end-point mappings (3) and apply

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our computations to compute control-variations for (1). Of course, the use of control-variations to study controllability is a standard tool in nonlinear controllability theory [8], [12], [4], [3], and their use can lead to valuable insight in the quest of narrowing the gap between the known sufficient and necessary Lie bracket conditions [12]. Moreover, and perhaps more importantly from a practical point of view, explicit constructions using end-point mappings can be used to construct locally asymptotically stabilizing piecewise analytic feedbacks for locally controllable systems [10]. We note that, although control-variations can be expressed in terms of Lie brackets via the Campbell–Baker–Hausdorff formula, we take the view point that for polynomial systems of the form (1), it is more natural to state controllability conditions in terms of the d-multilinear mapping $\mathbf{D}^d X_0(z_0)$ and its properties on the span$\{X_1(z_0),\ldots,X_m(z_0)\}$. An advantage in doing so is that the resulting conditions are invariant under (linear) feedback transformations, in contrast to Lie algebraic conditions which are known to depend on the specific choice of the system vector fields (see [13] for a simple physical system displaying this lack of invariance).

A. Notation

In this paper, all vector fields are assumed to be smooth, that is, infinitely differentiable. If $\boldsymbol{Z} = (Z_1,\ldots,Z_p)$ is a family of vector fields on $\mathbb{R}^n$, there exists an open set $\Omega_{\boldsymbol{Z}} \subseteq \mathbb{R}^p \times \mathbb{R}^n$ such that the mapping $\Phi^{\boldsymbol{Z}} : \Omega_{\boldsymbol{Z}} \to \mathbb{R}^n$ given by

$$\Phi^{\boldsymbol{Z}}(\boldsymbol{t},z) = \Phi^{Z_p}_{t_p} \circ \Phi^{Z_{p-1}}_{t_{p-1}} \circ \cdots \circ \Phi^{Z_1}_{t_1}(z)$$

is well-defined for all $(\boldsymbol{t},z) = (t_1,\ldots,t_p,z) \in \Omega_{\boldsymbol{Z}}$. To emphasize the dependence of $\Phi^{\boldsymbol{Z}}(\boldsymbol{t},z)$ on either $\boldsymbol{t}$ or z, at times it will be convenient to write $\Phi^{\boldsymbol{Z}}(\boldsymbol{t},z) = \Phi^{\boldsymbol{Z}}_{\boldsymbol{t}}(z) = \Phi^{\boldsymbol{Z}}_z(\boldsymbol{t})$, accepting a slight abuse of notation. The Lie bracket of vector fields Z_1 and Z_2 is denoted $[Z_1,Z_2]$ and is given by the formula $[Z_1,Z_2](z) = \mathbf{D}Z_2(z)Z_1(z) - \mathbf{D}Z_1(z)Z_2(z)$. The tangent space of $\mathbb{R}^n$ at z will be denoted as $T_z\mathbb{R}^n$.

For any set $P \subset \mathbb{R}^n$ we define $-P = \{-z \,:\, z \in P\}$ and by $\mathrm{conv}^+(P)$ we denote the convex cone generated by P, i.e., the set of all *positive* linear combinations of the elements of P. The positive integers are denoted $\mathbb{N} = \{1,2,\ldots,\}$ and the non-negative integers are denoted $\mathbb{N}_0 = \{0,1,2,\ldots\}$. We let $\mathbb{R}^p_{\geq 0} = \{(t_1,\ldots,t_p) \in \mathbb{R}^p \,:\, t_i \geq 0\}$. Finally, the symbol z_0 will be used exclusively to denote the zero vector in $\mathbb{R}^n$.

II. Homogeneous vector fields

This section is meant to be a review of basic facts regarding homogeneous polynomial vector fields and to establish some notation. Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. The d order derivative of X at $z \in \mathbb{R}^n$ is a symmetric d-multilinear mapping which we denote by $\mathbf{D}^d X(z) : (\mathbb{R}^n)^p \to \mathbb{R}^n$ and we will use the notation $\mathbf{D}^d X(z)(v_1,v_2,\ldots,v_d)$ to denote the evaluation of $\mathbf{D}^d X(z)$ at $v_1,\ldots,v_d \in \mathbb{R}^n$. If X is d-homogeneous, then $\mathbf{D}^d X(z) = \mathbf{D}^d X(\tilde{z})$ for all $z,\tilde{z}$ so that the point where the derivative is evaluated is immaterial and we will therefore omit it when no confusion arises. Now, if X is d-homogeneous and $z_0 = 0$, it is straightforward to show using the formula for the Lie bracket that

$$\mathbf{D}^d X(z_0)(v_1,\ldots,v_d) = [Y_1,[Y_2,[\cdots,[Y_d,X]]\cdots](z_0), \quad (4)$$

where $Y_1,\ldots,Y_d$ are arbitrary vector fields extending $v_1,\ldots,v_d$ at z_0, respectively, i.e., $Y_j(z_0) = v_j$. Of special interest is when $\mathbf{D}^d X(z_0)$ acts on the diagonal. Differentiating d times the relation $X(\lambda v) = \lambda^d X(v)$ with respect to λ and evaluating at $\lambda = 0$, we obtain that $\mathbf{D}^d X(z_0)(v,\ldots,v) = d!X(v)$. Therefore, if $Y(z_0) = v$ then

$$[Y,[Y,[\cdots,[Y,X]]\cdots](z_0) = d!X(v). \quad (5)$$

III. Odd Polynomial Systems

For purposes of exposition and to establish further notation, in this section we summarize the STLC property for systems of the form (1) in the case that X_0 is a homogeneous polynomial vector field of *odd* degree. The results in this section follow from the early work of Brunovský [5] on *odd systems* defined as follows (see also [15]). If $\mathcal{F}$ is a family of vector fields, its Lie closure is denoted by $\mathrm{Lie}(\mathcal{F})$ and $\mathrm{Lie}_z(\mathcal{F})$ denotes the evaluation of $\mathrm{Lie}(\mathcal{F})$ at z. A family of vector fields $\mathcal{F} = \{Z_i \,:\, i \in I\}$ on $\mathbb{R}^n$ is called *odd* on a symmetric neighbourhood $\Omega \subset \mathbb{R}^n$ of the origin if for every $i \in I$ there is a $j \in I$ such that $Z_j(-z) = -Z_i(z)$ for $z \in \Omega$. In [5] it is shown that if $\mathcal{F}$ is an odd family and satisfies the *Lie algebra rank condition* (LARC) at z_0, i.e., $\mathrm{Lie}_{z_0}(\mathcal{F}) = T_{z_0}\mathbb{R}^n$, then for each $T > 0$ the set of points reachable from z_0 in time T by following concatenations of the integral curves of the elements of $\mathcal{F}$ in forward time contains the origin in its interior. We remark that if the family $\mathcal{F}$ consists of real-analytic vector fields then it is well-known that the LARC condition at z_0 is also necessary for STLC from z_0 [14].

Consider now the polynomial system Σ given by (1) and let

$$\mathcal{F}_\Sigma = \{X_0 + \Sigma_{i=1}^m u_i X_i \,:\, u \in U\}.$$

If d is odd, then using the symmetry of U it is not hard to see that $\mathcal{F}_\Sigma$ is an odd family of vector fields on $\mathbb{R}^n$. Now since $\mathrm{Lie}(\mathcal{F}_\Sigma) = \mathrm{Lie}(\{X_0,X_1,\ldots,X_m\})$ and the vector fields $X_0,X_1,\ldots,X_m$ are real-analytic, it follows by [5] that Σ is STLC from z_0 if and only if $\mathrm{Lie}_{z_0}(\{X_0,X_1,\ldots,X_m\}) = T_{z_0}\mathbb{R}^n$, provided d is odd. Our purpose now is to relate the LARC condition at z_0 with the properties of $\mathbf{D}^d X_0(z_0)$ on the subspace $\mathrm{span}\{X_1(z_0),\ldots,X_m(z_0)\}$. To this end, it is shown in [2, Theorem 3.2] (see also [11, Lemma 5]) that $\mathrm{Lie}_{z_0}(\{X_0,X_1,\ldots,X_m\})$ in fact coincides with the smallest subspace containing the vectors $X_1(z_0),\ldots,X_m(z_0)$ and invariant under $\mathbf{D}^d X_0(z_0)$. In the following we summarize the constructions in [2] as they will be used in subsequent sections.

On the set of vector fields on $\mathbb{R}^n$, we define a non-associative $\mathbb{R}$-algebra structure by asking that

$$(YX)(z) = \mathbf{D}X(z)Y(z)$$

for vector fields X, Y. For example, from the chain-rule, a product of order three is

$$\begin{aligned}(ZYX)(z) &= \mathbf{D}(YX)(z)Z(z)\\ &= \mathbf{D}(\mathbf{D}XY)(z)Z(z)\\ &= \mathbf{D}^2X(z)(Y(z),Z(z)) + \mathbf{D}X(z)\mathbf{D}Y(z)Z(z).\end{aligned}$$

If $\mathcal{F}$ is a family of vector fields, we let $\mathrm{Alg}(\mathcal{F})$ denote the smallest subalgebra of vector fields which contains $\mathcal{F}$. It is clear that $\mathrm{Lie}(\mathcal{F}) \subseteq \mathrm{Alg}(\mathcal{F})$ for any family $\mathcal{F}$. Let now X_0 be a d-homogeneous vector field, let

$$\mathsf{B} = \mathrm{span}\{X_1(z_0),\ldots,X_m(z_0)\}$$

and denote by $\langle \mathbf{D}^d X_0; \mathsf{B}\rangle$ the smallest subspace containing B and invariant under $\mathbf{D}^d X_0$. A spanning set for $\langle \mathbf{D}^d X_0; \mathsf{B}\rangle$ can be constructed as follows. Let $S_0 = \{X_1(z_0),\ldots,X_m(z_0)\}$ and for $k \in \mathbb{N}$ define

$$S_k = S_{k-1} \cup \{\mathbf{D}^d X_0(v_1,\ldots,v_d) : v_i \in S_{k-1}\}.$$

Then there exists some $1 \le k \le n-m$ such that $\langle \mathbf{D}^d X_0; \mathsf{B}\rangle = \mathrm{span}(S_k)$. As shown in [2], the following equalities hold

$$\begin{aligned}\langle \mathbf{D}^d X_0; \mathsf{B}\rangle &= \mathrm{Alg}_{z_0}(\{X_0, X_1,\ldots,X_m\})\\ &= \mathrm{Lie}_{z_0}(\{X_0, X_1,\ldots,X_m\})\end{aligned}$$

and as a consequence we record the following theorem.

Theorem 3.1: Let Σ be a control-affine system of the form (1) *where X_0 is a d-homogeneous polynomial vector field and $X_1,\ldots,X_m$ are constant vector fields, and let $z_0 = 0 \in \mathbb{R}^n$. If d is an odd integer then Σ is STLC from z_0 if and only if $\langle \mathbf{D}^d X_0; \mathsf{B}\rangle = T_{z_0}\mathbb{R}^n$.*

Proof: If Σ is STLC then $\mathrm{Lie}_{z_0}(\{X_0, X_1,\ldots,X_m\}) = T_{z_0}\mathbb{R}^n$ because the vector fields $X_0, X_1,\ldots,X_m$ are real analytic. From $\mathrm{Lie}_{z_0}(\{X_0, X_1,\ldots,X_m\}) = \langle \mathbf{D}^d X_0; \mathsf{B}\rangle$ it follows that $\langle \mathbf{D}^d X_0; \mathsf{B}\rangle = T_{z_0}\mathbb{R}^n$. Conversely, if $\langle \mathbf{D}^d X_0; \mathsf{B}\rangle = T_{z_0}\mathbb{R}^n$ then also $\mathrm{Lie}_{z_0}(\{X_0, X_1,\ldots,X_m\}) = T_{z_0}\mathbb{R}^n$ and STLC of Σ then follows by [5] and the fact that $\mathcal{F}_\Sigma$ is an odd family of vector fields. ■

We remark that when d is even, Theorem 3.1 is no longer true. A trivial example is the planar system

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= x^2\end{aligned}$$

which satisfies the LARC at the origin but is clearly not STLC from the origin.

As stated in the introduction, our interest is in obtaining controllability conditions for polynomial systems of the form (1) in terms of the image of B under $\mathbf{D}^d X_0$. To this end, in the next section we introduce a class of high-order tangent vectors to the reachable set $\mathcal{R}_\Sigma(z_0, T)$.

IV. A class of variations

Given a family of vector fields $\boldsymbol{Z} = (Z_1,\ldots,Z_p)$ and a smooth mapping $\boldsymbol{\tau} : \mathbb{R} \to \mathbb{R}^p_{\ge 0}$, there is an $\epsilon > 0$ such that the curve $\Phi^{\boldsymbol{Z}}_{z_0} \circ \boldsymbol{\tau} : [0,\epsilon] \to \mathbb{R}^n$ given by

$$(\Phi^{\boldsymbol{Z}}_{z_0} \circ \boldsymbol{\tau})(s) = \Phi^{Z_p}_{\tau_p(s)} \circ \Phi^{Z_{p-1}}_{\tau_{p-1}(s)} \circ \cdots \circ \Phi^{Z_1}_{\tau_1(s)}(z_0)$$

is well-defined. The curve $s \mapsto (\Phi^{\boldsymbol{Z}}_{z_0} \circ \boldsymbol{\tau})(s)$ is based at z_0 at $s = 0$ and consists of points obtained by following concatenations of the integral curves of $Z_1,\ldots,Z_p$ in forward time. The *order* of the pair $(\boldsymbol{Z}, \boldsymbol{\tau})$ at z_0 is the smallest integer $k \ge 1$ such that

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}\right|_{s=0} \Phi^{\boldsymbol{Z}}_{z_0}(\boldsymbol{\tau}(s)) \neq 0,$$

provided such an integer exists, and in this case we set

$$V_{\boldsymbol{Z},\boldsymbol{\tau}} := \left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}\right|_{s=0} \Phi^{\boldsymbol{Z}}_{z_0}(\boldsymbol{\tau}(s)).$$

Now consider the control system Σ given by (1) and let $\mathcal{V}^k_\Sigma$ denote for each positive integer k the set of all $V_{\boldsymbol{Z},\boldsymbol{\tau}}$ obtained by taking $\boldsymbol{Z} = (Z_1,\ldots,Z_p) \subset \mathcal{F}_\Sigma$, for all $p \ge 1$, and let

$$\mathcal{V}_\Sigma = \bigcup_{k\ge 1} \mathcal{V}^k_\Sigma.$$

By definition, $\mathcal{V}_\Sigma$ is a set of high-order tangent vectors at z_0 to the reachable set $\mathcal{R}_\Sigma(z_0, T)$. We list the main properties of $\mathcal{V}_\Sigma$.

Proposition 4.1 ([1], [12]): The following hold:

(i) *For each $k \ge 1$, the set $\mathcal{V}^k_\Sigma$ is a convex cone.*

(ii) *For integers $\ell, k \ge 1$, $\mathcal{V}^k_\Sigma \subset \mathcal{V}^{k\ell}_\Sigma$.*

(iii) *Properties (i) and (ii) imply that $\mathcal{V}_\Sigma$ is a convex cone.*

Proposition 4.1 and a degree theory argument can be used to prove the following (see [1], [12]).

Theorem 4.1: If $\mathcal{V}_\Sigma = T_{z_0}\mathbb{R}^n$ then Σ is STLC from z_0.

A complete understanding of the set of variations $\mathcal{V}_\Sigma$ for general control-affine systems remains a challenging open problem. For the case of polynomial control-affine systems of the form (1), our intent is to explicitly compute a class of these variations to obtain a sufficient condition for STLC. These variations will be computed by studying the Taylor series coefficients of the end-point maps $\boldsymbol{t} \mapsto \Phi^{\boldsymbol{Z}}_{z_0}(\boldsymbol{t})$, and their connection with labeled rooted trees, by borrowing ideas from Butcher [7]. As shown in [7], the Taylor series coefficients of an integral curve of a vector field Z can be expressed as a sum over so-called *elementary differentials* associated to Z. In our case, we are considering concatenations of the integral curves of a collection of vector fields, whereas in [7] only a single vector field is considered. Therefore, for our purposes we need to make a slight extension to the original definition of an elementary differential given in [7, pg. 151]. To this end and for later use, we introduce some terminology from graph theory related to rooted trees following closely the notation in [7, pg. 137-138].

Let σ be a *rooted tree*, i.e., a connected graph with no cycles and with a distinguished vertex called the *root*. There is a natural ordering imposed on the vertices of σ by declaring that the maximum vertex be the root of σ and if v_1 and v_2 are vertices of σ then $v_1 \le v_2$ if v_2 is in the unique path from the root of σ to v_1. If $v_1 \le v_2$ and v_1 and v_2 form an edge, then v_1 is the *child* of v_2 and v_2 is the *parent* of v_1. A vertex of a rooted tree without children is called a *leaf*. Now let $S \subset \mathbb{N}$ be a finite subset containing $|S|$ elements. Let T_S^* denote the set of rooted trees of order $|S|$ whose vertices are labeled with the elements of S such that the root is labeled $\max(S)$ and such that for each $a \in S \backslash \{\max(S)\}$ the labels of the vertices on the unique path from the root to the vertex labeled a forms a decreasing sequence. If $\sigma \in T_S^*$ and $\sigma_1, \sigma_2, \ldots, \sigma_d$ denote the rooted trees obtained by removing the root of σ and its incident edges, then the labeling of the vertices of σ induces a set partition $S_1, S_2, \ldots, S_d$ of $S \backslash \{\max(S)\}$ satisfying $\sigma_i \in T_{S_i}^*$ for $i = 1, \ldots, d$. For this reason, it is convenient to denote such a rooted tree σ using the notation $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_d]$. We now make the following extension of the definition of an elemantary differential given in [7, pg. 151].

Definition 4.1: Let $S = \{a_1, \ldots, a_k\} \subset \mathbb{N}$ and let $\sigma = [\sigma_1, \ldots, \sigma_d] \in T_S^*$. Let $S_1, \ldots, S_d$ denote the partition of $S \backslash \{\max(S)\}$ induced by the labeling of σ. For a family $\boldsymbol{Z} = \{Z_{a_1}, \ldots, Z_{a_k}\}$ of smooth vector fields on $\mathbb{R}^n$ the ***labeled elementary differential of Z corresponding to*** *σ is the map $\boldsymbol{Z}_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\boldsymbol{Z}_\sigma(z) = Z_{a_1}(z)$, if $S = \{a_1\}$, and*

$$\boldsymbol{Z}_\sigma(z) = (\mathbf{D}^d Z_{\max(S)}(z))(\boldsymbol{Z}_{\sigma_1}(z), \ldots, \boldsymbol{Z}_{\sigma_d}(z)),$$

if $|S| \ge 2$, where $\sigma_i \in T_{S_i}^*$ for $i = 1, \ldots, d$.
The following examples illustrate the previous definition.

Example 4.1: Let $S = \{4, 5, 6, 7, 12\}$ and consider the left-most labeled rooted tree $\sigma \in T_S^*$ shown in Fig. 1. Then with $\boldsymbol{Z} = \{Z_4, Z_5, Z_6, Z_7, Z_{12}\}$ one can verify by definition that

$$\boldsymbol{Z}_\sigma = \mathbf{D}^2 Z_{12}(\mathbf{D}Z_7 Z_4, \mathbf{D}Z_6 Z_5)$$

□

Example 4.2: As another example, suppose that $S = \{1, 3, 5, 7, 8, 11, 15, 18, 21\}$ and let $\sigma \in T_S^*$ denote the right-most labeled rooted tree shown in Fig. 1. Then with $\boldsymbol{Z} = \{Z_1, Z_3, Z_5, Z_7, Z_8, Z_{11}, Z_{15}, Z_{18}, Z_{21}\}$ one can verify that

$$\boldsymbol{Z}_\sigma = \mathbf{D}Z_{21}\mathbf{D}^2 Z_{18}(\mathbf{D}^3 Z_8(Z_7, Z_3, Z_1), \mathbf{D}Z_{15}\mathbf{D}Z_{11}Z_5)$$

□

Having introduced labeled elementary differentials, we now give a combinatorial description of the Taylor series coefficients of the maps $\boldsymbol{t} \mapsto \Phi_{z_0}^{\boldsymbol{Z}}(\boldsymbol{t})$. To this end, for a given family of vector fields $\boldsymbol{Z} = (Z_1, \ldots, Z_p)$ on $\mathbb{R}^n$ and a multi-index $I = (i_1, \ldots, i_p) \in (\mathbb{N}_0)^p$ we write

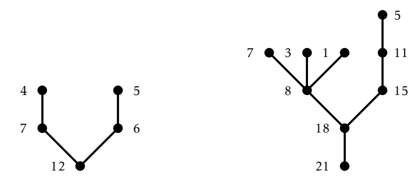

Fig. 1. Labeled rooted trees for Examples 4.1 and 4.2.

$\boldsymbol{Z}^I = Z_1^{i_1} Z_2^{i_2} \cdots Z_p^{i_p}$ for

$$\underbrace{Z_1 \cdots Z_1}_{i_1\text{-times}} \underbrace{Z_2 \cdots Z_2}_{i_2\text{-times}} \cdots \underbrace{Z_p \cdots Z_p}_{i_p\text{-times}}.$$

With this notation, it is not hard to show that the Taylor series expansion of $\Phi_{z_0}^{\boldsymbol{Z}} : \mathbb{R}^p \to \mathbb{R}^n$ about the origin in $\mathbb{R}^p$ is given by

$$\sum_{\ell=0}^{\infty} \sum_{|I|=\ell} (\boldsymbol{Z}^I)(z_0) \frac{\boldsymbol{t}^I}{I!} \tag{6}$$

where $|I| = i_1 + \cdots + i_p$, $I! = i_1! i_2! \cdots i_p!$, and $\boldsymbol{t}^I = t_1^{i_1} t_2^{i_2} \cdots t_p^{i_p}$ for $\boldsymbol{t} = (t_1, \ldots, t_p) \in \mathbb{R}^p$. If $|I| = \ell$ and we let $W_j = Z_1$ for $j = 1, \ldots, i_1$, $W_{i_1+j} = Z_2$, for $j = 1, \ldots, i_2$, etc., and finally let $W_{\ell - i_p + j} = Z_p$ for $j = 1, \ldots, i_p$, it is clear that

$$\boldsymbol{Z}^I(z_0) = (Z_1^{i_1} Z_2^{i_2} \cdots Z_p^{i_p})(z_0) = (W_1 W_2 \cdots W_\ell)(z_0).$$

Hence, in order to understand the combinatorial structure of the Taylor series coefficients $\boldsymbol{Z}^I(z_0)$ we need only understand the combinatorial structure of products of the form $(W_1 W_2 \cdots W_\ell)(z_0)$. This was done by J.C. Butcher [7, pg. 153-154] in the study of high-order Runge-Kutta methods for ordinary differential equations, and we summarize his findings in the following theorem.

Theorem 4.2 ([7]): Let $\mathbf{W} = (W_1, \ldots, W_\ell)$ be a family of smooth vector fields on $\mathbb{R}^n$ and let $S = \{1, \ldots, \ell\}$. Then

$$(W_1 W_2 \cdots W_\ell)(z) = \sum_{\sigma \in T_S^*} \mathbf{W}_\sigma(z).$$

for all $z \in \mathbb{R}^n$.

We illustrate Theorem 4.2 with the following example.

Example 4.3: Let $S = \{1, 2, 3, 4\}$ and let $W_1, \ldots, W_4$ be smooth vector fields on $\mathbb{R}^n$. Then $W_3 W_4 = \mathbf{D}W_4 W_3$ and therefore, by the chain-rule

$$\begin{aligned} W_2 W_3 W_4 &= \mathbf{D}(W_3 W_4)(W_2) \\ &= \mathbf{D}(\mathbf{D}W_4 W_3)(W_2) \\ &= \mathbf{D}^2 W_4(W_3, W_2) + \mathbf{D}W_4 \mathbf{D}W_3 W_2. \end{aligned}$$

Therefore, by the chain-rule, $W_1 W_2 W_3 W_4$ is equal to

$$\begin{aligned} &\mathbf{D}^3 W_4(W_3, W_2, W_1) + \mathbf{D}^2 W_4(\mathbf{D}W_3 W_1, W_2) \\ &+ \mathbf{D}^2 W_4(W_3, \mathbf{D}W_2 W_1) + \mathbf{D}^2 W_4(\mathbf{D}W_3 W_2, W_1) \\ &+ \mathbf{D}W_4 \mathbf{D}^2 W_3(W_2, W_1) + \mathbf{D}W_4 \mathbf{D}W_3 \mathbf{D}W_2 W_1. \end{aligned} \tag{7}$$

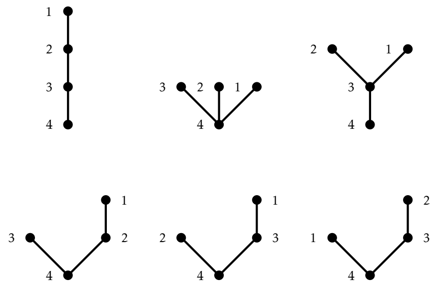

Fig. 2. The elements in the set T_S^* for $S = \{1,2,3,4\}$.

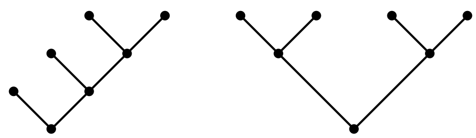

Fig. 3. The only binary rooted trees of order $\ell = 7$.

On the other hand, there are a total of six elements in the set T_S^*, as shown in Fig. 2. By inspection, there is a one-to-one correspondence with each $\sigma \in T_S^*$ and each elementary differential appearing in (7). □

Theorem 4.2 can be used to obtain a key simplification in the structure of the Taylor series expansion of $\boldsymbol{t} \mapsto \Phi_{z_0}^{\boldsymbol{Z}}$ when $\boldsymbol{Z} \subset \mathcal{F}_\Sigma$ and where Σ is a control-affine system (1) with an d-homogeneous polynomial drift vector field X_0 and constant control-input vector fields. In the following, by a *proper d-ary tree* we mean a rooted tree in which every vertex that is not a leaf has exactly d children. For example, the only proper 2-ary trees, i.e., proper binary trees, having $\ell = 7$ vertices are displayed in Fig. 3. We now state the following useful result whose proof is omitted due to space limitations.

Lemma 4.1: Let $\ell \geq 2$ be an integer and let $\boldsymbol{W} = (W_1, \ldots, W_\ell)$ be a family of smooth vector fields such that $W_j = X_0 + Y_j$, where X_0 is a d-homogeneous polynomial vector field with $d \geq 2$ and Y_j are non-zero constant vector fields. Let $S = \{1, \ldots, \ell\}$ and let $\rho_d(T_S^*)$ denote the subset of proper d-ary trees contained in T_S^*. If $\sigma \in T_S^*$ is not a proper d-ary tree then $\boldsymbol{W}_\sigma(z_0) = 0$. Consequently

$$(W_1 W_2 \cdots W_\ell)(z_0) = \sum_{\sigma \in \rho_d(T_S^*)} \boldsymbol{W}_\sigma(z_0). \tag{8}$$

In the next section, Lemma 4.1 will be used to determine the order of a pair $(\boldsymbol{Z}, \boldsymbol{\tau})$ and the explicit expression for the resulting tangent vector $V_{\boldsymbol{Z},\boldsymbol{\tau}}$. This is, in general, a difficult and tedious task and is usually done by using the Campbell–Baker–Hausdorff (CBH) formula, see for instance [9]. On the other hand, it is the opinion of the author that the combinatorial nature of Lemma 4.1 could alleviate this difficulty, especially at high-orders where the CBH formula becomes intractable.

V. Main Theorem

Consider the control-affine system (1) with X_0 a d-homogeneous polynomial vector field and let $Y_u = \sum_{i=1}^m u_i X_i$ for $u \in U$. Let $P_0 = \{Y_u(z_0) \ : \ u \in U\}$ and iteratively define the sets

$$P_{k+1} = \{\mathbf{D}^d X_0(v, \ldots, v) \ : \ \pm v \in P_k\} \tag{9}$$

for $k \in \mathbb{N}_0$. The main result of this paper is the following theorem.

Theorem 5.1: Consider the polynomial control-affine system

$$\Sigma_q : \begin{cases} \dot{x} = u \\ \dot{y} = q_1(x) + q_2(y) \end{cases}$$

where the state variable is $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^r$, $u \in \mathbb{R}^m$ is the control variable, and $q_1 : \mathbb{R}^m \to \mathbb{R}^r$ and $q_2 : \mathbb{R}^r \to \mathbb{R}^r$ are homogeneous quadratic maps. Define the sets P_{k+1} as in (9). If

$$\mathrm{conv}^+(P_0) + \mathrm{conv}^+(P_1) + \mathrm{conv}^+(P_2) = \mathbb{R}^m \times \mathbb{R}^r \tag{10}$$

then Σ_q is STLC from $z_0 = (0,0) \in \mathbb{R}^m \times \mathbb{R}^r$.

We give only a sketch of the proof due to space limitations.

Sketch of proof to Theorem 5.1. Let us denote by X_0 the drift vector field and $X_1, \ldots, X_m$ the control-input vector fields defined by the control-affine system Σ_q. For $u \in U$ let $Z_u = X_0 + Y_u$ where we recall that $Y_u = \sum_{i=1}^m u_i X_i$.

First, it is clear that

$$\mathcal{V}_\Sigma^1 = \mathrm{conv}^+(P_0).$$

Next, a direct computation using (6) gives that

$$\Phi_s^{Z_{-u}} \circ \Phi_s^{Z_u}(z_0) = 2\mathbf{D}^2 X_0(Y_u(z_0), Y_u(z_0)) \frac{s^3}{3!} + o(s^3)$$

and therefore $2\mathbf{D}^2 X_0(Y_u(z_0), Y_u(z_0)) \in \mathcal{V}_\Sigma^3$. By Proposition 4.1, we conclude that $\mathbf{D}^2 X_0(Y_u(z_0), Y_u(z_0)) \in \mathcal{V}_\Sigma^3$, and since $u \in U$ was arbitrary and $P_0 = -P_0$ by symmetry of U, it follows that $P_1 \subseteq \mathcal{V}_\Sigma^3$. Therefore $\mathrm{conv}^+(P_1) \subseteq \mathcal{V}_\Sigma^3$.

Now suppose that $\pm v \in P_1$. By definition, there exists $u, \bar{u} \in U$ such that $v = \mathbf{D}^2 X_0(Y_u, Y_u)$ and $-v = \mathbf{D}^2 X_0(Y_{\bar{u}}, Y_{\bar{u}})$, and therefore,

$$\Phi_s^{Z_{-u}} \circ \Phi_s^{Z_u}(z_0) = 2v \frac{s^3}{3!} + o(s^3)$$
$$\Phi_s^{Z_{-\bar{u}}} \circ \Phi_s^{Z_{\bar{u}}}(z_0) = -2v \frac{s^3}{3!} + o(s^3).$$

Consider the curve

$$\alpha(s) = \Phi_s^{Z_{-\bar{u}}} \Phi_s^{Z_{\bar{u}}} \Phi_s^{Z_{-u}} \Phi_s^{Z_u}(z_0).$$

By construction [1], the derivatives of α at $s = 0$ of orders $1, 2, 3$ all vanish, so the first possibly non-zero derivative of α at $s = 0$ is of order $\ell = 4$. From Lemma 4.1, the only possibly non-zero elementary differentials contributing to each term of the Taylor series of α are those associated with proper binary trees. There are no proper binary trees of orders $\ell = 4$ or $\ell = 6$,

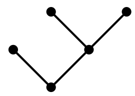

Fig. 4. The only binary rooted trees of order $\ell = 5$.

and thus $\alpha^{(4)}(0) = \alpha^{(6)}(0) = 0$. A direct computation of the derivative $\alpha^{(5)}(0)$ shows that it depends on the *mixed* derivatives of $X_0(x,y) = (0, q_1(x) + q_2(y))$, that is, on derivatives that depend explicitly on both x and y. Since X_0 has no mixed terms in x and y, $\alpha^{(5)}(0) = 0$. One then computes, using the combinatorial formula (8), that $\alpha^{(7)}(0)$ is given explicitly by

$$\alpha^{(7)}(0) = 460\mathbf{D}^2 X_0(\mathbf{D}^2 X_0(Y_u, Y_u), \mathbf{D}^2 X_0(Y_u, Y_u)).$$

Hence $\mathbf{D}^2 X_0(v,v) \in \mathcal{V}_\Sigma^7$ and since $v \in P_1$ was arbitrary this shows that $P_2 \subset \mathcal{V}_\Sigma^7$ and therefore $\mathrm{conv}^+(P_2) \subset \mathcal{V}_\Sigma^7$. By definition $\mathcal{V}_\Sigma^k \subset \mathcal{V}_\Sigma$ for all $k \in \mathbb{N}$, and therefore

$$\mathrm{conv}^+(P_0) + \mathrm{conv}^+(P_1) + \mathrm{conv}^+(P_2) \subseteq \mathcal{V}_\Sigma.$$

Assumption (10) and Theorem 4.1 completes the proof. ■

We now give an example of the applicability of Theorem 5.1 when known sufficient conditions fail, e.g. [15, Theorem 7.3].

Example 5.1: Consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_3 \\ \dot{y}_1 &= 2y_1^2 - y_2^2 - y_3^2 + x_1^2 - x_2^2 \\ \dot{y}_2 &= y_1^2 - 2y_2^2 + x_1^2 - x_2^2 \\ \dot{y}_3 &= y_2^2 - y_3^2 + x_2^2 - 2x_3^2 \end{aligned} \tag{11}$$

so that $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, and

$$\begin{aligned} q_1(x) &= (x_1^2 - x_2^2, x_1^2 - x_2^2, x_2^2 - 2x_3^2) \\ q_2(y) &= (2y_1^2 - y_2^2 - y_3^2, y_1^2 - 2y_2^2, y_2^2 - y_3^2). \end{aligned}$$

Let X_0, X_1, X_2, X_3 denote the system vector fields defined by (11) and let $U = [-1,1]^3$ so that U is symmetric. Let $u = (1,0,0)$, $\bar{u} = (0,1,\frac{1}{\sqrt{2}})$, and $\tilde{u} = (0,0,\frac{1}{\sqrt{2}})$, and $u' = (1,1,0)$. Now, recalling that $\mathbf{D}^2 X_0(Y_u, Y_u) = 2! X_0(Y_u)$, we have that

$$\begin{aligned} \mathbf{D}^2 X_0(Y_u, Y_u) &= (0,0,0,2,2,0) \\ \mathbf{D}^2 X_0(Y_{\bar{u}}, Y_{\bar{u}}) &= (0,0,0,-2,-2,0) \\ \mathbf{D}^2 X_0(Y_{\tilde{u}}, Y_{\tilde{u}}) &= (0,0,0,0,0,-2) \\ \mathbf{D}^2 X_0(Y_{u'}, Y_{u'}) &= (0,0,0,0,0,2) \end{aligned}$$

Therefore,

$$\pm(0,0,0,2,2,0), \pm(0,0,0,0,0,2) \in P_1.$$

Now let $v_1 = (0,0,0,2,2,0)$ and $v_2 = (0,0,0,0,0,2)$. Then

$$\begin{aligned} \mathbf{D}^2 X_0(v_1, v_1) &= (0,0,0,8,-8,8) \\ \mathbf{D}^2 X_0(v_2, v_2) &= (0,0,0,-8,0,-8) \end{aligned}$$

and therefore

$$v_3 = (0,0,0,8,-8,8), \quad v_4 = (0,0,0,-8,0,-8) \in P_2.$$

One can verify that $\{\pm X_1, \pm X_2, \pm X_3, \pm v_1, \pm v_2, v_3, v_4\}$ forms a positive basis for $\mathbb{R}^6$ and therefore condition (10) is satisfied. By Theorem 5.1, (11) is STLC from the origin $z_0 = 0 \in \mathbb{R}^6$.

On the other hand, the system in consideration fails to satisfy [15, Theorem 7.3]. Indeed, according to [15, Theorem 7.3], the Lie bracket

$$\beta = [X_1,[X_1,X_0]] + [X_2,[X_2,X_0]] + [X_3,[X_3,X_0]]$$

must be expressible at z_0 as a linear combination of lower order brackets. One directly computes that $\beta(z_0) = (0,0,0,0,0,-2)$. It is clear that any bracket of order lower than β when evaluated at z_0 belongs to the subspace $\mathrm{span}\{X_1(z_0), X_2(z_0), X_3(z_0)\}$ since $[X_i, X_0](z_0) = 0$ for any $i \in \{1,2,3\}$. Clearly, $\beta(z_0) \notin \mathrm{span}\{X_1(z_0), X_2(z_0), X_3(z_0)\}$ and so one cannot deduce STLC for this system from [15, Theorem 7.3]. □

References

[1] C.O. Aguilar, A.D. Lewis, *Small-time local controllability for a class of homogeneous systems*, SIAM Journal on Control and Optimization, Vol. 50, No. 3, pp. 1502–1517, 2012.

[2] J. Baillieul, *The geometry of homogeneous polynomial dynamical systems*, Nonlinear Analysis. Theory, Methods & Applications, 4 (1980), pp. 879–900.

[3] R. Bianchini, *Variations of a control process at the initial point*, Journal of Optimization Theory and Applications, Vol. 81, No. 2, pp. 249-258, 1994.

[4] R. Bianchini, G. Stefani, *Controllability along a trajectory: A variational approach*, SIAM Journal on Control and Optimization, 31 (1993), pp. 900–927.

[5] P. Brunovský, *Local controllability of odd systems*, Banach Center Publications, 1 (1976), pp. 39–46.

[6] F. Bullo, A.D. Lewis, *Geometric Control of Mechanical Systems*, Springer-Verlag, New York-Heiderlberg-Berlin, 2004.

[7] J. Butcher, *Numerical Methods for Ordinary Differential Equations*, Second Edition, John Wiley & Sons Ltd, 2008.

[8] H. Frankowska, *Local controllability of control systems with feedback*, Journal of Optimization Theory and Applications, Vol. 60, No. 2, pp. 277–296, 1989.

[9] H. Hermes, *Lie algebras of vector fields and local approximation of attainable sets*, SIAM Journal of Control and Optimization, Vol. 16, No. 5, pp. 715–727, 1978.

[10] P. Isaiah, *Feedback stabilisation of locally controllable systems*, Proc. 50th IEEE Conf. Decision and Control, pp. 1074–1079, 2011.

[11] V. Jurdjevic and I. Kupka, *Polynomial control systems*, Mathematische Annalen, 272 (1985), pp. 361–368.

[12] M. Kawski, *High-order small-time local controllability*, in Nonlinear Controllability and Optimal Control, H. Sussmann, ed., Marcel Dekker, Inc., 1990, pp. 431–467.

[13] J. Melody, T. Başar, and F. Bullo, *On nonlinear controllability of homogeneous systems linear in control*, IEEE Transactions on Automatic Control, 48 (2003), pp. 139–143.

[14] H. Sussmann, V. Jurdjevic, *Controllability of nonlinear systems*, Journal of Differential Equations, Vol. 12, pp.95–116, 1972.

[15] H. Sussmann, *A general theorem on local controllability*, SIAM Journal on Control and Optimization, 25 (1987), pp. 158–194.