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Spectral characterizations of anti-regular graphs

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ABSTRACT

We study the eigenvalues of the unique connected anti-regular graph A_n. Using Chebyshev polynomials of the second kind, we obtain a trigonometric equation whose roots are the eigenvalues and perform elementary analysis to obtain an almost complete characterization of the eigenvalues. In particular, we show that the interval $\Omega = [\frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}]$ contains only the trivial eigenvalues $\lambda = -1$ or $\lambda = 0$, and any closed interval strictly larger than Ω will contain eigenvalues of A_n for all n sufficiently large. We also obtain bounds for the maximum and minimum eigenvalues, and for all other eigenvalues we obtain interval bounds that improve as n increases. Moreover, our approach reveals a more complete picture of the bipartite character of the eigenvalues of A_n, namely, as n increases the eigenvalues are (approximately) symmetric about the number $-\frac{1}{2}$. We also obtain an asymptotic distribution of the eigenvalues as $n \to \infty$. Finally, the relationship between the eigenvalues of A_n and the eigenvalues of a general threshold graph is discussed.

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1. Introduction

Let $G = (V, E)$ be an n-vertex simple graph, that is, a graph without loops or multiple edges, and let $\deg_G(v)$ denote the degree of $v \in V$. It is an elementary exercise to show that G contains at least two vertices of equal degree. If G has all vertices with equal degree then G is called a *regular* graph. We say then that G is an *anti-regular* graph if G has only two vertices of equal degree. If G is anti-regular it follows easily that the complement graph $\overline{G}$ is also anti-regular since $\deg_G(v) = (n-1) - \deg_{\overline{G}}(v)$. It was shown in [2] that up to isomorphism, there is only one connected anti-regular graph on n vertices and that its complement is the unique disconnected n-vertex anti-regular graph. Let us denote by A_n the unique connected anti-regular graph on $n \geq 2$ vertices. The graph A_n has several interesting properties. For instance, it was shown in [3] that A_n is *universal for trees*, that is, every tree graph on n vertices is isomorphic to a subgraph of A_n. Anti-regular graphs are *threshold graphs* [4] which have numerous applications in computer science and psychology. Within the family of threshold graphs, the anti-regular graph is uniquely defined by its *independence polynomial* [7]. Also, the eigenvalues of the Laplacian matrix of A_n are all distinct integers and the missing eigenvalue from $\{0, 1, \ldots, n\}$ is $\lfloor (n+1)/2 \rfloor$. In [6], the characteristic and matching polynomial of A_n are studied and several recurrence relations are obtained for these polynomials, along with some spectral properties of the adjacency matrix of A_n.

In this paper, we study the eigenvalues of the adjacency matrix of A_n. If $V(G) = \{v_1, \ldots, v_n\}$ is the vertex set of the graph G then the adjacency matrix of G is the $n \times n$ symmetric matrix A with entry $A(i,j) = 1$ if v_i and v_j are adjacent and $A(i,j) = 0$ otherwise. From now on, whenever we refer to the eigenvalues of a graph we mean the eigenvalues of its adjacency matrix. It is known that the eigenvalues of A_n have algebraic multiplicity equal to one and take on a bipartite character [6] in the sense that if n is even then half of the eigenvalues are negative and the other half are positive, and if n is odd then $\lambda = 0$ is an eigenvalue and half of the remaining eigenvalues are positive and the other half are negative. Our approach to studying the eigenvalues of A_n relies on a natural labeling of the vertices that results in a block triangular structure for the inverse adjacency matrix. The blocks are tridiagonal pseudo-Toeplitz matrices and Hankel matrices. We are then able to employ the connection between tridiagonal Toeplitz matrices and Chebyshev polynomials to obtain a trigonometric equation whose roots are the eigenvalues. Performing elementary analysis on the roots of the equation we obtain an almost complete characterization of the eigenvalues of A_n. In particular, we show that the only eigenvalues contained in the closed interval $\Omega = [\frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}]$ are the trivial eigenvalues $\lambda = -1$ or $\lambda = 0$, and any closed bounded interval strictly larger than Ω will contain eigenvalues of A_n for all n sufficiently large. This improves a result in [10] obtained for general threshold graphs and we conjecture that Ω is a forbidden eigenvalue interval for all threshold graphs (besides the trivial eigenvalues $\lambda = 0$ or $\lambda = -1$). We also obtain bounds for the maximum and minimum eigenvalues, and for all other eigenvalues we obtain interval bounds that improve as n increases. Moreover, our approach reveals a more complete picture of the bipartite character of the eigenvalues of A_n, namely, as n increases the non-trivial eigenvalues are (approximately) symmetric about the number $-\frac{1}{2}$. Lastly, we obtain an asymptotic distribution of the eigenvalues as $n \to \infty$. We conclude the paper by arguing that a characterization of the eigenvalues of A_n will shed light on the broader problem of characterizing the spectrum of general threshold graphs.

2. Main results

It is known that the eigenvalues of A_n are simple and that $\lambda = -1$ is an eigenvalue if n is even and $\lambda = 0$ is an eigenvalue if n is odd [6]. In either case, we will call $\lambda = -1$ or $\lambda = 0$ the *trivial eigenvalue* of A_n and will be denoted by λ_0. Throughout this paper, we denote the positive eigenvalues of A_n as

$$\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+$$

and the negative eigenvalues (excluding λ_0) as

$$\lambda_{k-1}^- < \lambda_{k-2}^- < \cdots < \lambda_1^-$$

if $n = 2k$ is even and

$$\lambda_k^- < \lambda_{k-1}^- < \cdots < \lambda_1^-$$

if $n = 2k + 1$ is odd. The eigenvalues are labeled this way because $\{\lambda_j^+, \lambda_j^-\}$ should be thought of as a pair for $j \in \{1, 2, \ldots, k-1\}$. In [10], it is proved that a threshold graph has no eigenvalue in the interval $(-1, 0)$. Our first result supplies a forbidden interval for the non-trivial eigenvalues of A_n.

Theorem 2.1. *Let A_n denote the connected anti-regular graph with n vertices. The only eigenvalue of A_n in the interval $\Omega = [\frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}]$ is $\lambda_0 \in \{-1, 0\}$.*

Based on numerical experimentation, and our observations in Section 8, we make the following conjectures.

Conjecture 2.1. For any n, the anti-regular graph A_n has the smallest positive eigenvalue and has the largest non-trivial negative eigenvalue among all threshold graphs on n vertices.

By Theorem 2.1, a proof of the previous conjecture would also prove the following.

Conjecture 2.2. Other than the trivial eigenvalues $\{0, -1\}$, the interval $\Omega = [\frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}]$ does not contain an eigenvalue of any threshold graph.

Our next result establishes the asymptotic behavior of the eigenvalues of smallest magnitude as $n \to \infty$.

Theorem 2.2. *Let A_n be the connected anti-regular graph with $n = 2k$ if n is even and $n = 2k + 1$ if n is odd. Let $\lambda_1^+(k)$ denote the smallest positive eigenvalue of A_n and let $\lambda_1^-(k)$ denote the negative eigenvalue of A_n closest to the trivial eigenvalue λ_0. The following hold:*

(i) *The sequence $\{\lambda_1^+(k)\}_{k=1}^{\infty}$ is strictly decreasing and converges to $\frac{-1+\sqrt{2}}{2}$.*
(ii) *The sequence $\{\lambda_1^-(k)\}_{k=1}^{\infty}$ is strictly increasing and converges to $\frac{-1-\sqrt{2}}{2}$.*

As a result, the interval $\Omega = [\frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}]$ in Theorem 2.1 is best possible in the sense that any closed bounded interval strictly larger than Ω will contain eigenvalues of A_n (other than the trivial eigenvalue) for all sufficiently large n.

Our next main result says that $\lambda_j^+ + \lambda_j^- + 1 \approx 0$ for almost all $j \in \{1, 2, \ldots, k-1\}$ provided that k is sufficiently large. In other words, the eigenvalues are approximately symmetric about the number $-\frac{1}{2}$.

Theorem 2.3. *Let A_n be the connected anti-regular graph where $n = 2k$ or $n = 2k + 1$. Fix $r \in (0, 1)$ and let $\varepsilon > 0$ be arbitrary. Then for k sufficiently large,*

$$|\lambda_j^+ + \lambda_j^- + 1| < \varepsilon$$

for all $j \in \{1, 2, \ldots, k-1\}$ such that $\frac{2j}{2k-1} \le r$ if n is even and $\frac{j}{k} \le r$ if n is odd.

Note that the proportion of integers $j \in \{1, 2, \ldots, k-1\}$ that satisfy the inequality in Theorem 2.3 is r. Hence, Theorem 2.3 implies that as k increases a larger proportion of the eigenvalues are (approximately) symmetric about the point $-\frac{1}{2}$. Lastly, we obtain an asymptotic distribution of the eigenvalues of all anti-regular graphs.

Theorem 2.4. *Let $\sigma(n)$ denote the set of the eigenvalues of A_n, let $\sigma = \bigcup_{n\ge1} \sigma(n)$, and let $\bar{\sigma}$ denote the closure of σ. Then*

$$\bar{\sigma} = (-\infty, \tfrac{-1-\sqrt{2}}{2}] \cup \{0, -1\} \cup [\tfrac{-1+\sqrt{2}}{2}, \infty).$$

It turns out that if we restrict n to even then $\bar{\sigma} = (-\infty, \frac{-1-\sqrt{2}}{2}] \cup \{-1\} \cup [\frac{-1+\sqrt{2}}{2}, \infty)$, and if we restrict n to odd then $\bar{\sigma} = (-\infty, \frac{-1-\sqrt{2}}{2}] \cup \{0\} \cup [\frac{-1+\sqrt{2}}{2}, \infty)$.

3. Eigenvalues of tridiagonal Toeplitz matrices

Our study of the eigenvalues of A_n relies on the relationship between the eigenvalues of tridiagonal Toeplitz matrices and Chebyshev polynomials [8,9], and so we briefly review the necessary background. The *Chebyshev polynomial of the second kind* of degree m, denoted by $U_m(x)$, is the unique polynomial such that

$$U_m(\cos\theta) = \frac{\sin((m+1)\theta)}{\sin(\theta)}. \tag{1}$$

The first several U_m's are $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, and $U_3(x) = 8x^3 - 4x$. The sequence of polynomials $\{U_m\}_{m=0}^{\infty}$ satisfies the three-term recurrence relation

$$U_m(x) = 2xU_{m-1}(x) - U_{m-2}(x) \tag{2}$$

for $m \geq 2$. From (1), the zeros $x_1, x_2, \ldots, x_m$ of $U_m(x)$ are easily determined to be

$$x_j = \cos\left(\frac{j\pi}{m+1}\right), \quad j = 1, 2, \ldots, m.$$

Chebyshev polynomials are used extensively in numerical analysis and differential equations and the reader is referred to [8] for a thorough introduction to these interesting polynomials.

A real tridiagonal Toeplitz matrix is a matrix of the form

$$T = \begin{pmatrix} a & c & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & c \\ & & b & a \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. For our purposes, and to simplify the presentation, we assume that $c = b$. We can then write $T = aI + bM$ where I is the identity matrix and

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}.$$

If λ is an eigenvalue of M then clearly $a + b\lambda$ is an eigenvalue of T. Let $\phi_m(t) = \det(tI - M)$ denote the characteristic polynomial of the $m \times m$ matrix M. The Laplace expansion of $\phi_m(t)$ along the last row produces the recurrence relation

$$\phi_m(t) = t\phi_{m-1}(t) - \phi_{m-2}(t)$$

for $m \geq 2$, with $\phi_0(t) = 1$ and $\phi_1(t) = t$. It then follows that $\phi_m(t) = U_m(t/2)$. Indeed, we have that $U_0(t/2) = 1$ and $U_1(t/2) = 2(t/2) = t$, and from the recurrence (2) we have

$$U_m(t/2) = 2(t/2)U_{m-1}(t/2) - U_{m-2}(t/2) = tU_{m-1}(t/2) - U_{m-2}(t/2).$$

4. The anti-regular graph A_n

As already mentioned, the anti-regular graph A_n is an example of a threshold graph. Threshold graphs were first studied independently by Chvátal and Hammer [11] and by Henderson and Zalcstein [12]. There exists an extensive literature on the applications and algorithmic aspects of threshold graphs and the reader is referred to [4,5] for a thorough introduction. A threshold graph G on $n \geq 2$ vertices can be obtained via an iterative procedure as follows. One begins with a single vertex v_1 and at step $i \geq 2$ a new vertex v_i is added that is either connected to all existing vertices (a dominating vertex) or not connected to any of the existing vertices (an isolated vertex). The iterative construction of G is best encoded with a *binary creation sequence* $b = (b_1, b_2, \ldots, b_n)$ where $b_1 = 0$ and, for $i \in \{2, \ldots, n\}$, $b_i = 1$ if v_i was added as a dominating vertex or $b_i = 0$ if v_i was added as an isolated vertex. The resulting vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ that is consistent with the iterative construction of G will be called the *canonical labeling* of G. In the canonical labeling, the adjacency matrix of G takes the form

$$A = \begin{pmatrix} 0 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ b_2 & 0 & b_3 & \cdots & \vdots & \vdots \\ b_3 & b_3 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & b_{n-1} & \vdots \\ b_{n-1} & \cdots & \cdots & b_{n-1} & 0 & b_n \\ b_n & \cdots & \cdots & \cdots & b_n & 0 \end{pmatrix}. \tag{3}$$

For the anti-regular graph A_n, the associated binary sequence is $b = (0, 1, 0, 1, \ldots, 0, 1)$ if n is even and is $b = (0, 0, 1, 0, 1, \ldots, 0, 1)$ if n is odd. In what follows, we focus on the case that n is even. In Section 7, we describe the details for the case that n is odd.

Example 4.1. When $n = 8$ the graph A_n in the canonical labeling is shown in Fig. 1 and the associated adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

As will be seen, a distinct labeling of the vertex set of A_n results in a block structure for A. Let $J = J_k$ denote the $k \times k$ all ones matrix and let $I = I_k$ denote the $k \times k$ identity matrix.

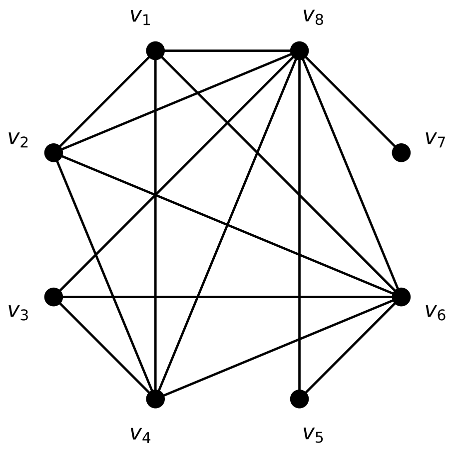

Fig. 1. The connected anti-regular graph A_8 in the canonical labeling.

Lemma 4.1. *The adjacency matrix of A_{2k} can be written as*

$$A = \begin{pmatrix} 0 & B \\ B & J - I \end{pmatrix} \tag{4}$$

where B is the $k \times k$ Hankel matrix

$$B = \begin{pmatrix} & & & & 1 \\ & & & \iddots & 1 \\ & & \iddots & \iddots & \vdots \\ & \iddots & \iddots & & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

Proof. Recall that A_n is the unique connected graph on n vertices that has exactly only two vertices of the same degree. Moreover, it is known [2] that the repeated degree of A_{2k} is k, that is, the degree sequence of A_{2k} in non-increasing order is

$$d(A_n) = \left(n-1, n-2, \ldots, \frac{n}{2}, \frac{n}{2}, \frac{n}{2} - 1, \ldots, 2, 1\right). \tag{5}$$

It is clear that the degree sequence of the graph with adjacency matrix (4) is also (5). Since A_n is uniquely determined by its degree sequence the claim holds. □

Remark 4.1. Starting with the canonically labelled vertex set of A_n, the permutation

$$\sigma = \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_{n-2} & v_{n-1} & v_n \\ v_{\frac{n}{2}} & v_{\frac{n}{2}+1} & v_{\frac{n}{2}-1} & \cdots & v_{n-1} & v_1 & v_n \end{pmatrix} \tag{6}$$

relabels the vertices of A_n so that its adjacency matrix is transformed from (3) to (4) via the permutation matrix associated to σ. The newly labelled graph is such that $\deg(v_i) \leq \deg(v_{i+1})$. For example, when $n = 8$ the adjacency matrix (4) is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

To study the eigenvalues of A we will obtain an eigenvalue equation for A^{-1}. Expressions for A^{-1} involving sums of certain matrices are known when the vertex set of A_n is canonically labelled [1]. On the other hand, our choice of vertex labels for A_n produces a closed-form expression for A^{-1}. The proof of the following is left as a straightforward computation.

Lemma 4.2. *Consider the adjacency matrix (4) of A_n where $n = 2k$. Then*

$$A^{-1} = \begin{pmatrix} V & W \\ W & 0 \end{pmatrix}$$

where $W = B^{-1}$ and $V = -B^{-1}(J - I)B^{-1}$. Explicitly,

$$W = \begin{pmatrix} & & & -1 & 1 \\ & & \iddots & \iddots & \\ & \iddots & \iddots & & \\ -1 & \iddots & & & \\ 1 & & & & \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 0 \end{pmatrix}.$$

Notice that W is a Hankel matrix and the $(k-1) \times (k-1)$ leading principal submatrix of V is a tridiagonal Toeplitz matrix.

Example 4.2. For our running example when $n = 8$ we have

$$A^{-1} = \begin{pmatrix} V & W \\ W & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 2 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. The eigenvalues of A_n

Suppose that $z = (x, y) \in \mathbb{R}^{2k}$ is an eigenvector of A^{-1} with eigenvalue $\alpha \in \mathbb{R}$, where $x, y \in \mathbb{R}^k$. From $A^{-1}z = \alpha z$ we obtain the two equations

$$\begin{aligned} Vx + Wy &= \alpha x \\ Wx &= \alpha y \end{aligned}$$

and after substituting $y = \frac{1}{\alpha} Wx$ into the first equation and re-arranging we obtain

$$(\alpha^2 I - \alpha V - W^2)x = 0.$$

Clearly, we must have $x \neq 0$. Let $R(\alpha) = \alpha^2 I - \alpha V - W^2$ so that $\det(R(t)) = \det(tI - A^{-1})$ is the characteristic polynomial of A^{-1}. It is straightforward to verify that

$$R(\alpha) = \begin{pmatrix} f(\alpha) & \alpha+1 & & & \\ \alpha+1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & f(\alpha) & \alpha+1 \\ & & & \alpha+1 & \alpha^2-1 \end{pmatrix}$$

where $f(\alpha) = \alpha^2 - 2\alpha - 2$. Since it is already known that $\alpha = -1$ is an eigenvalue of A^{-1} (this can easily be seen from the last column or row of $R(\alpha)$), we consider instead the matrix

$$S(\alpha) = \frac{1}{(\alpha+1)} R(\alpha) = \begin{pmatrix} h(\alpha) & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & h(\alpha) & 1 \\ & & & 1 & \alpha-1 \end{pmatrix}$$

where $h(\alpha) = \frac{\alpha^2 - 2\alpha - 2}{\alpha+1}$. Hence, $\alpha \neq -1$ is an eigenvalue of A^{-1} if and only if $\det(S(\alpha)) = 0$. We now obtain a recurrence relation for $\det(S(\alpha))$. To that end, notice that the $(k-1) \times (k-1)$ leading principal submatrix of $S(\alpha)$ is a tridiagonal Toeplitz matrix. Hence, for $m \geq 1$ define

$$\phi_m(\alpha) = \det \begin{pmatrix} h(\alpha) & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & h(\alpha) \end{pmatrix}_{m \times m}.$$

A straightforward Laplace expansion of $\det(S(\alpha))$ along the last row yields

$$\det(S(\alpha)) = (\alpha - 1)\phi_{k-1}(\alpha) - \phi_{k-2}(\alpha).$$

Hence, $\alpha \neq -1$ is an eigenvalue of A^{-1} if and only if

$$(\alpha - 1)\phi_{k-1}(\alpha) - \phi_{k-2}(\alpha) = 0.$$

On the other hand, for $m \geq 2$ the Laplace expansion of $\phi_m(\alpha)$ along the last row produces the recurrence relation

$$\phi_m(\alpha) = h(\alpha)\phi_{m-1}(\alpha) - \phi_{m-2}(\alpha)$$

with $\phi_0(\alpha) = 1$ and $\phi_1(\alpha) = h(\alpha)$. We can therefore conclude that $\phi_m(\alpha) = U_m\left(\frac{h(\alpha)}{2}\right)$ and thus $\alpha \neq -1$ is an eigenvalue of A^{-1} if and only if

$$(\alpha - 1)U_{k-1}\left(\frac{h(\alpha)}{2}\right) - U_{k-2}\left(\frac{h(\alpha)}{2}\right) = 0. \tag{7}$$

Substituting $\alpha = \frac{1}{\lambda}$ into (7) and re-arranging yields

$$\lambda = \frac{U_{k-1}(\beta(\lambda))}{U_{k-1}(\beta(\lambda)) + U_{k-2}(\beta(\lambda))}$$

where $\beta(\lambda) = \frac{h(1/\lambda)}{2} = \frac{1-2\lambda-2\lambda^2}{2\lambda(\lambda+1)}$. Recalling the definition (1) of $U_m(x)$, we have proved the following.

Theorem 5.1. *Let $n = 2k$ and let A_n denote the connected anti-regular graph with n vertices. Then λ is an eigenvalue of A_n if and only if*

$$\lambda = \frac{\sin(k\theta)}{\sin(k\theta) + \sin((k-1)\theta)} \tag{8}$$

where $\theta = \arccos\left(\frac{1-2\lambda-2\lambda^2}{2\lambda(\lambda+1)}\right)$.

Remark 5.1. In [6, Theorem 3], recurrence relations for the characteristic polynomial of the adjacency matrix of A_n involving Chebyshev polynomials are obtained using combinatorial methods.

We now analyze the character of the solution set of (8). To that end, first define the function

$$\theta(\lambda) = \arccos\left(\frac{1-2\lambda-2\lambda^2}{2\lambda(\lambda+1)}\right).$$

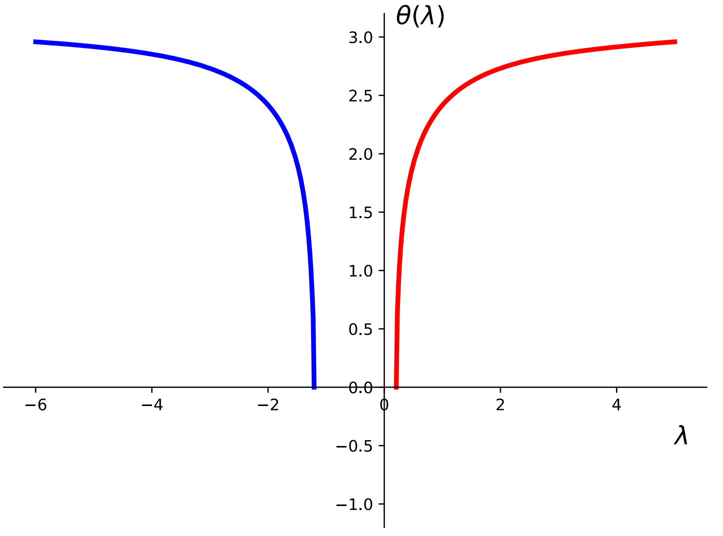

Fig. 2. Graph of the function $\theta(\lambda) = \arccos\left(\frac{1-2\lambda^2-2\lambda}{2\lambda(\lambda+1)}\right)$ on its domain $(-\infty, \frac{-1-\sqrt{2}}{2}] \cup [\frac{-1+\sqrt{2}}{2}, \infty)$.

Using the fact that the domain and range of arccos is $[-1, 1]$ and $[0, \pi]$, respectively, it is straightforward to show that the domain and range of $\theta(\lambda)$ is $(-\infty, \frac{-1-\sqrt{2}}{2}] \cup [\frac{-1+\sqrt{2}}{2}, \infty)$ and $[0, \pi)$, respectively. The graph of $\theta(\lambda)$ is displayed in Fig. 2. Next, define the function

$$F(\theta) = \frac{\sin(k\theta)}{\sin(k\theta) + \sin((k-1)\theta)}. \tag{9}$$

In the interval $(0, \pi)$, the function F has vertical asymptotes at

$$\gamma_j = \frac{2j\pi}{2k-1}, \quad j = 1, 2, \ldots, k-1.$$

This follows from the trigonometric identity

$$\sin(k\theta) + \sin((k-1)\theta) = 2\sin\frac{(2k-1)\theta}{2}\cos\frac{\theta}{2}.$$

For notational consistency we define $\gamma_0 = 0$. Hence, F is continuously differentiable on the set $(0, \gamma_1) \cup (\gamma_1, \gamma_2) \cup \cdots (\gamma_k, \pi)$. Moreover, using l'Hópital's rule it is straightforward to show that

$$\lim_{\theta \to 0} F(\theta) = \frac{k}{2k-1}$$

and

$$\lim_{\theta \to \pi} F(\theta) = k.$$

Hence, there is no harm in defining $F(0) = \frac{k}{2k-1}$ and $F(\pi) = k$ so that we can take $D = [0, \gamma_1) \cup (\gamma_1, \gamma_2) \cup \cdots \cup (\gamma_{k-1}, \pi]$ as the domain of continuity of F.

We can now prove Theorem 2.1.

Proof of Theorem 2.1. The domain of $\theta(\lambda)$ does not contain any point in the interior of Ω and therefore no solution of (8) is in the interior of Ω. At the boundary points of Ω we have

$$\theta(\tfrac{-1-\sqrt{2}}{2}) = \theta(\tfrac{-1+\sqrt{2}}{2}) = 0.$$

On the other hand, $F(0) = \frac{k}{2k-1}$ and thus the boundary points of Ω are not solutions to (8) either. The case that n is odd is similar and will be dealt with in Section 7. □

We now analyze solutions to (8) by treating θ as the unknown variable and expressing λ in terms of θ. To that end, solving for λ from the equation $\theta = \arccos\left(\frac{1-2\lambda^2-2\lambda}{2\lambda(\lambda+1)}\right)$ yields the two solutions

$$\lambda = f_1(\theta) = \frac{-(\cos\theta + 1) + \sqrt{(\cos\theta+1)(\cos\theta+3)}}{2(\cos\theta+1)}$$
$$\lambda = f_2(\theta) = \frac{-(\cos\theta + 1) - \sqrt{(\cos\theta+1)(\cos\theta+3)}}{2(\cos\theta+1)}. \tag{10}$$

Notice that

$$f_1(\theta) + f_2(\theta) = -1, \tag{11}$$

a fact that will be used to show the bipartite character of large anti-regular graphs. Both f_1 and f_2 are continuous on $[0, \pi)$, continuously differentiable on $(0, \pi)$, and $\lim_{\theta\to\pi^-} f_1(\theta) = \infty$ and $\lim_{\theta\to\pi^-} f_2(\theta) = -\infty$. In Figs. 3–4, we plot the functions $f_1(\theta), f_2(\theta)$, and $F(\theta)$ for the values $k = 8$ and $k = 16$ in the interval $0 \le \theta \le \pi$. A dashed line at the value $\lambda = -\frac{1}{2} = \frac{f_1(\theta)+f_2(\theta)}{2}$ is included to emphasize that it is a line of symmetry between the graphs of f_1 and f_2.

Figs. 3–4 show that the graphs of F and f_1 intersect exactly k times, say at $\theta_1^+, \ldots, \theta_k^+$, and thus $\lambda_j^+ = f_1(\theta_j^+)$ for $j = 1, 2, \ldots, k$ are the positive eigenvalues of A_n. Similarly, F and f_2 intersect exactly $(k-1)$ times, say at $\theta_1^-, \ldots, \theta_{k-1}^-$, and thus $\lambda_j^- = f_2(\theta_j^-)$ for $j = 1, 2, \ldots, k-1$ are the negative eigenvalues of A_n besides the eigenvalue $\lambda = -1$. The following theorem formalizes the above observations and supplies interval estimates for the eigenvalues.

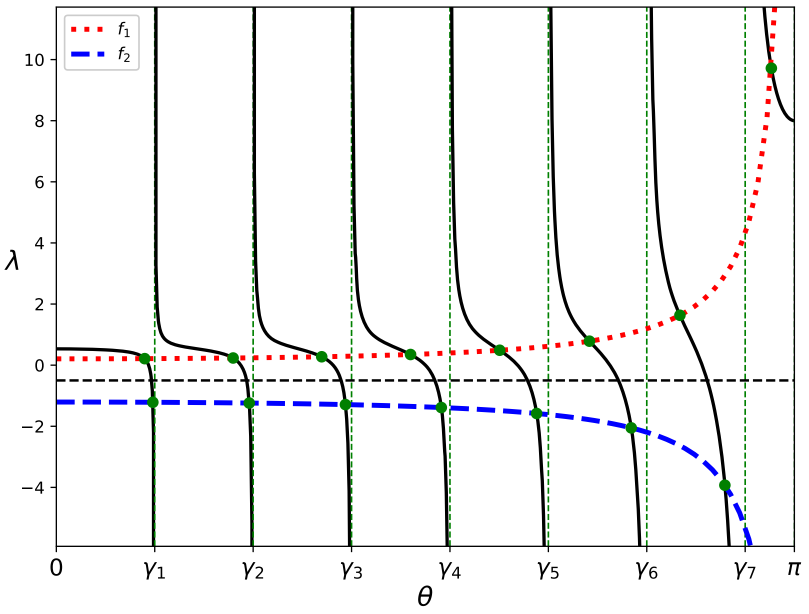

Fig. 3. Graph of the functions $f_1(\theta), f_2(\theta)$, and $F(\theta)$ (black) for $\theta \in [0, \pi]$ for $k = 8$.

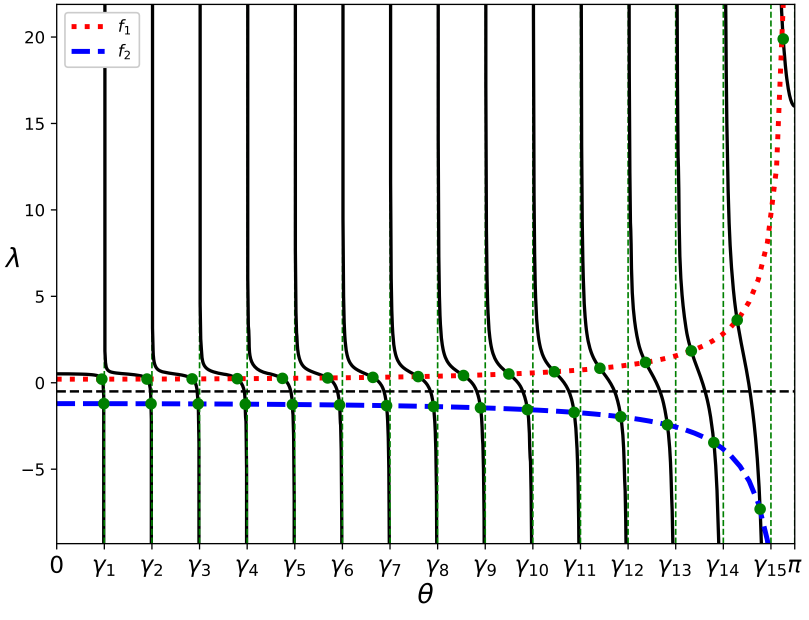

Fig. 4. Graph of the functions $f_1(\theta), f_2(\theta)$, and $F(\theta)$ (black) for $\theta \in [0, \pi]$ for $k = 16$.

Theorem 5.2. *Let A_n be the connected anti-regular graph with $n = 2k$ vertices. Let $F(\theta)$ be defined as in (9) and let $f_1(\theta)$ and $f_2(\theta)$ be defined as in (10), and recall that $\gamma_j = \frac{2\pi j}{2k-1}$ for $j = 0, 1, \ldots, k-1$.*

(i) *The functions $F(\theta)$ and $f_1(\theta)$ intersect exactly k times in the interval $0 < \theta < \pi$. If $\theta_1^+ < \theta_2^+ < \cdots < \theta_k^+$ are the intersection points then the positive eigenvalues of A_n are*

$$f_1(\theta_1^+) < f_1(\theta_2^+) < \cdots < f_1(\theta_k^+).$$

Moreover, for $j = 1, 2, \ldots, k-1$ it holds that

$$f_1(\gamma_{j-1}) < f_1(\theta_j^+) < f_1(\gamma_j).$$

(ii) *The functions $F(\theta)$ and $f_2(\theta)$ intersect exactly $(k-1)$ times in the interval $0 < \theta < \pi$. If $\theta_1^- < \theta_2^- < \cdots < \theta_{k-1}^-$ are the intersection points then the negative eigenvalues of A_n are*

$$f_2(\theta_{k-1}^-) < \cdots < f_2(\theta_2^-) < f_2(\theta_1^-) < -1.$$

Moreover, for $j = 1, 2, \ldots, k-1$ it holds that

$$f_2(\gamma_j) < f_2(\theta_j^-) < f_2(\gamma_{j-1}).$$

Proof. One computes that

$$f_1'(\theta) = \frac{\sin\theta}{2(\cos\theta + 1)\sqrt{(\cos\theta + 1)(\cos\theta + 3)}}$$

and thus $f_1'(\theta) > 0$ for $\theta \in (0, \pi)$. Therefore, f_1 is strictly increasing on the interval $(0, \pi)$. Since $f_2(\theta) = -1 - f_1(\theta)$ it follows that f_2 is strictly decreasing on the interval $(0, \pi)$. On the other hand, using basic trigonometric identities and the relation $\sin(\theta)U_{k-1}(\cos\theta) = \sin(k\theta)$, we compute that

$$F'(\theta) = \frac{[-k + U_{k-1}(\cos\theta)\cos((k-1)\theta)]\sin\theta}{[\sin(k\theta) + \sin((k-1)\theta)]^2}.$$

It is known that $\max_{x\in[-1,1]} |U_m(x)| = (m + 1)$ and the maximum occurs at $x = \pm 1$ [8]. Therefore, $F'(\theta) < 0$ for all $\theta \in D\backslash\{0, \pi\}$. It follows that F is a strictly decreasing function on D, and when restricted to the interval (γ_j, γ_{j+1}) for any $j = 1, \ldots, k-2$, F is a bijection onto $(-\infty, \infty)$. Now, since f_1 is a strictly increasing continuous function on $[\gamma_j, \gamma_{j+1}]$ for $j = 1, 2, \ldots, k-2$, the graphs of F and f_1 intersect at exactly one point inside the interval (γ_j, γ_{j+1}). A similar argument applies to f_2 and F on each interval (γ_j, γ_{j+1}) for $j = 1, 2, \ldots, k-2$. Now consider the leftmost interval $[0, \gamma_1)$. We have that

Table 1
The ratio $t_k = \frac{(\theta_k^+ - \gamma_{k-1})}{(\pi - \gamma_{k-1})}$ for $k = 125, 250, 500, \ldots, 32000$.

$n = 2k$	t_k
250	0.5020031290
500	0.5010007838
1000	0.5005001962
2000	0.5002500492
4000	0.5001250123
8000	0.5000625018
16000	0.5000312567
32000	0.5000156204

$f_1(0) < F(0)$ and since f_1 is strictly increasing and continuous on $[0, \gamma_1]$, and F is strictly decreasing and $\lim_{\theta \to \gamma_1^-} F(\theta) = -\infty$, F and f_1 intersect only once in the interval $(0, \gamma_1)$. A similar argument holds for f_2 and F on the interval $(0, \gamma_1)$. Finally, on the interval $(\gamma_{k-1}, \pi]$, we have $f_2(\gamma_{k-1}) < F(\pi)$ and since f_2 decreases and F is strictly increasing on the interval (γ_{k-1}, π) then f_2 and F do not intersect there. On the interval (γ_{k-1}, π), f_1 has vertical asymptote at $\theta = \pi$ and is strictly increasing and F is continuous and decreasing on $(\gamma_{k-1}, \pi]$. Thus, in $(\gamma_{k-1}, \pi]$, f_1 and F intersect only once. This completes the proof. □

Theorem 2.2 now follows from the fact that $\lim_{k\to\infty} f_1(\theta_1^+) = f_1(0) = \frac{-1+\sqrt{2}}{2}$ and that $\lim_{k\to\infty} f_2(\theta_1^-) = f_2(0) = \frac{-1-\sqrt{2}}{2}$. We also obtain the following corollary.

Corollary 5.1. *Let $\lambda_{\max} > 0$ and $\lambda_{\min} < 0$ denote the largest and smallest eigenvalues, respectively, of the connected anti-regular graph A_n where n is even. Then*

$$F(\pi) = \frac{n}{2} < \lambda_{\max}$$

and

$$f_2\left(\tfrac{2(n/2-1)\pi}{n-1}\right) < \lambda_{\min}.$$

Through numerical experiments, we have determined that the mid-point of the interval (γ_{k-1}, π), which is $\frac{(4k-3)\pi}{2(2k-1)}$, is a good approximation to $\theta_k^+ \in (\gamma_{k-1}, \pi)$, that is,

$$\lambda_{\max} \approx F\left(\frac{(4k-3)\pi}{2(2k-1)}\right).$$

In Table 1 we show the results of computing the ratio $t_k = \frac{(\theta_k^+ - \gamma_{k-1})}{(\pi - \gamma_{k-1})}$ for $k = 125, 250, \ldots, 32000$ which shows that possibly $\lim_{k\to\infty} t_k = \frac{1}{2}$.

6. The eigenvalues of large anti-regular graphs

A graph G is called *bipartite* if there exists a partition $\{X, Y\}$ of the vertex set $V(G)$ such that any edge of G contains one vertex in X and the other in Y. It is known that the eigenvalues of a bipartite graph G are symmetric about the origin. Figs. 3–4 reveal that for the connected anti-regular graph A_{2k} a similar symmetry property about the point $-\frac{1}{2}$ is approximately true. Specifically, if $\lambda \neq \lambda_{\max}$ is a positive eigenvalue of A_{2k} then $-1-\lambda$ is approximately an eigenvalue of A_{2k}, and moreover the proportion $r \in (0,1)$ of the eigenvalues that satisfy this property to within a given error $\varepsilon > 0$ increases as the number of vertices increases.

Recall that if $\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+$ denote the positive eigenvalues of A_{2k} then there exists unique $\theta_1^+ < \theta_2^+ < \cdots < \theta_k^+$ in the interval $(0, \pi)$ such that $\lambda_j^+ = f_1(\theta_j^+)$, and if $\lambda_{k-1}^- < \lambda_{k-2}^- < \cdots < \lambda_1^- < -1$ denote the negative eigenvalues of A_{2k} there exists unique $\theta_1^- < \theta_2^- < \cdots < \theta_{k-1}^-$ in $(0, \pi)$ such that $\lambda_j^- = f_2(\theta_j^-)$ for $j = 1, 2, \ldots, k-1$. With this notation we now prove Theorem 2.3.

Proof of Theorem 2.3. Both $f_1(\theta)$ and $f_2(\theta)$ are continuous on $[0, \pi)$ and therefore are uniformly continuous on the interval $[0, r\pi]$. Hence, there exists $\delta > 0$ such that if $\theta, \gamma \in [0, r\pi]$ and $|\theta - \gamma| < \delta$ then $|f_1(\theta) - f_1(\gamma)| < \varepsilon/2$ and $|f_2(\theta) - f_2(\gamma)| < \varepsilon/2$. Let k be such that $\frac{2\pi}{2k-1} \leq \delta$ and let $j^* \in \{1, \ldots, k-1\}$ be the largest integer such that $\frac{2j^*}{2k-1} \leq r$. Then for all $j \in \{1, \ldots, j^*\}$ it holds that $[\gamma_{j-1}, \gamma_j] \subset [0, r\pi]$. Let $c_j \in [\gamma_{j-1}, \gamma_j]$ be arbitrarily chosen for each $j \in \{1, \ldots, j^*\}$. Then $\theta_j^+, \theta_j^-, c_j \in [\gamma_{j-1}, \gamma_j]$ implies that $|f_1(\theta_j^+) - f_1(c_j)| < \varepsilon/2$ and $|f_2(\theta_j^-) - f_2(c_j)| < \varepsilon/2$ for $j \in \{1, \ldots, j^*\}$. Therefore, if $j \in \{1, \ldots, j^*\}$ then

$$\begin{aligned}
|\lambda_j^+ + \lambda_j^- + 1| &= |f_1(\theta_j^+) + f_2(\theta_j^-) + 1| \\
&= |f_1(\theta_j^+) - f_1(c_j) + f_1(c_j) + f_2(\theta_j^-) - f_2(c_j) + f_2(c_j) + 1| \\
&\leq |f_1(\theta_j^+) - f_1(c_j)| + |f_2(\theta_j^-) - f_2(c_j)| + |f_1(c_j) + f_2(c_j) + 1| \\
&= |f_1(\theta_j^+) - f_1(c_j)| + |f_2(\theta_j^-) - f_2(c_j)| \\
&< \varepsilon
\end{aligned}$$

where we used the fact that $f_1(c_j) + f_2(c_j) + 1 = 0$. This completes the proof for the even case. As discussed in Section 7, the odd case is similar. □

Note that the proportion of $j \in \{1, 2, \ldots, k-1\}$ such that $\frac{2j}{2k-1} \leq r$ is approximately r. In the next theorem we obtain estimates for $|\lambda_j^+ + \lambda_j^- + 1|$ using the Mean Value theorem.

Theorem 6.1. *Let A_n be the connected anti-regular graph where $n = 2k$. Then for all $1 \le j \le k-1$ it holds that*

$$|\lambda_j^+ + \lambda_j^- + 1| \le \frac{4\pi f_1'(\gamma_j)}{2k-1}.$$

In particular, for fixed $r \in (0,1)$ and a given arbitrary $\varepsilon > 0$, if k is such that $\frac{4\pi f_1'(r\pi)}{2k-1} < \varepsilon$ then

$$|\lambda_j^+ + \lambda_j^- + 1| < \varepsilon$$

for all $1 \le j \le \frac{(2k-1)r}{2}$.

Proof. First note that since $f_1(\theta) + f_2(\theta) = -1$ it follows that $f_2'(\theta) = -f_1'(\theta)$. The derivative f_1' vanishes at $\theta = 0$, is non-negative and strictly increasing on $[0, \pi)$. Therefore, by the Mean value theorem, on any closed interval $[a, b] \subset [0, \pi)$, both $f_1(\theta)$ and $f_2(\theta)$ are Lipschitz with constant $K = f_1'(b)$. Hence, a similar computation as in the proof of Theorem 2.3 shows that

$$|\lambda_j^+ + \lambda_j^- + 1| \le \frac{4\pi f_1'(\gamma_j)}{2k-1}$$

for $j = 1, 2, \ldots, k-1$. Therefore, if k is such that $\frac{4\pi f_1'(r\pi)}{2k-1} < \varepsilon$ then for $1 \le j \le \frac{(2k-1)r}{2}$ we have that $\frac{2\pi j}{2k-1} \le r\pi$ and therefore

$$|\lambda_j^+ + \lambda_j^- + 1| \le \frac{4\pi f_1'(\gamma_j)}{2k-1} \le \frac{4\pi f_1'(r\pi)}{2k-1} < \varepsilon. \quad \square$$

A similar proof gives the following estimates for the eigenvalues with error bounds.

Theorem 6.2. *Let A_n be the connected anti-regular graph where $n = 2k$. For $1 \le j \le k-1$ it holds that*

$$|\lambda_j^+ - f_1(\gamma_j)| \le \frac{2\pi f_1'(\gamma_j)}{2k-1}$$

and

$$|\lambda_j^- - f_2(\gamma_j)| \le \frac{2\pi f_1'(\gamma_j)}{2k-1}.$$

We now prove Theorem 2.4.

Proof of Theorem 2.4. It is clear that $\{-1, 0\} \subset \sigma \subset \bar{\sigma}$. Let $\varepsilon > 0$ be arbitrary and let $y \in [\frac{-1+\sqrt{2}}{2}, \infty)$. Then $y \in \bar{\sigma}$ if there exists $\mu \in \sigma$ such that $|\mu - y| < \varepsilon$. If $y \in \sigma$ the result is trivial, so assume that $y \notin \sigma$. Since $f_1 : [0, \pi) \to [\frac{-1+\sqrt{2}}{2}, \infty)$ is a bijection, there exists a unique $\theta' \in [0,\pi)$ such that $y = f_1(\theta')$. Let $c \in [0,\pi)$ be such that $\theta' < c < \pi$. For k sufficiently large, there exists $j \in \{1,\dots,k-1\}$ such that $\theta' \in [\gamma_{j-1},\gamma_j]$ and $\frac{2j\pi}{2k-1} \le c$. Increasing k if necessary, we can ensure that also $\frac{2\pi f_1'(c)}{2k-1} < \varepsilon$. Then by the Mean value theorem applied to f_1 on the interval $I = [\min\{\theta',\theta_j^+\},\max\{\theta',\theta_j^+\}]$, there exists $c_j \in I$ such that

$$|\lambda_j^+ - y| = |f_1(\theta_j^+) - f_1(\theta')| \le |\theta_j^+ - \theta'||f_1'(c_j)| < \frac{2\pi}{2k-1} f_1'(c) < \varepsilon,$$

where in the penultimate inequality we used the fact that f_1' is increasing and $c_j < c$. This proves that y is a limit point of σ and thus $y \in \bar{\sigma}$. A similar argument can be performed in the case that $y \in (-\infty, \frac{-1-\sqrt{2}}{2}]$ using f_2. $\square$

7. The odd case

In this section, we give an overview of the details for the case that A_n is the unique connected anti-regular graph with $n = 2k+1$ vertices. In the canonical labeling of A_n, the partition $\pi = \{\{v_1,v_2\},\{v_3\},\{v_4\},\dots,\{v_n\}\} = \{C_1,C_2,\dots,C_{2k}\}$ is an *equitable partition* of A_n [14]. In other words, π is the *degree partition* of A_n (we note that this is true for any threshold graph). The quotient graph A_n/π has vertex set π and its $2k \times 2k$ adjacency matrix is

$$A/\pi = \begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 2 & 0 & 0 & 1 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 1 & \cdots & \vdots & \vdots \\ 2 & 1 & 1 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 2 & 1 & 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

In other words, A/π is obtained from the adjacency matrix of the anti-regular graph A_{2k} (in the canonical labeling) with the 1's in the first column replaced by 2's. It is a standard result that all of the eigenvalues of A/π are eigenvalues of A_n [14]. At this point, we proceed just as in Section 4. Under the same permutation (6) of the vertices of A_n/π, the quotient adjacency matrix A/π takes the block form

$$A/\pi = \begin{pmatrix} 0 & B \\ C & J-I \end{pmatrix}$$

where

$$C = \begin{pmatrix} & & & & 2 \\ & & & 1 & 2 \\ & & \iddots & \vdots & \vdots \\ & \iddots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

Then

$$(A/\pi)^{-1} = \begin{pmatrix} -C^{-1}(J-I)B^{-1} & C^{-1} \\ B^{-1} & 0 \end{pmatrix}.$$

After computations similar to the even case, the analogue of (7) is

$$\frac{(\alpha^2 - 1/2)}{\alpha + 1} U_{k-1}\left(\frac{h(\alpha)}{2}\right) - \frac{1}{2} U_{k-2}\left(\frac{h(\alpha)}{2}\right) = 0.$$

After making the substitution $\alpha = \frac{1}{\lambda}$ and simplifying one obtains

$$\frac{(2-\lambda^2)}{\lambda(\lambda+1)} U_{k-1}(\beta(\lambda)) - U_{k-2}(\beta(\lambda)) = 0$$

or equivalently

$$\frac{(2-\lambda^2)}{\lambda(\lambda+1)} = \frac{U_{k-2}(\beta(\lambda))}{U_{k-1}(\beta(\lambda))} = \frac{\sin((k-1)\theta)}{\sin(k\theta)}.$$

The analogue of Theorem 5.1 in the odd case is the following.

Theorem 7.1. *Let $n = 2k+1$ and let A_n denote the connected anti-regular graph with n vertices. Then $\lambda \neq 0$ is an eigenvalue of A_n if and only if*

$$\frac{(2-\lambda^2)}{\lambda(\lambda+1)} = \frac{\sin((k-1)\theta)}{\sin(k\theta)} \tag{12}$$

where $\theta = \arccos\left(\frac{1-2\lambda-2\lambda^2}{2\lambda(\lambda+1)}\right)$.

Define the function $g(\lambda) = \frac{(2-\lambda^2)}{\lambda(\lambda+1)}$. Changing variables from λ to θ as in the even case, and defining $g_1(\theta) = g(f_1(\theta))$, $g_2(\theta) = g(f_2(\theta))$, and in this case $F(\theta) = \frac{\sin((k-1)\theta)}{\sin(k\theta)}$, we obtain the two equations

$$g_1(\theta) = F(\theta)$$
$$g_2(\theta) = F(\theta).$$

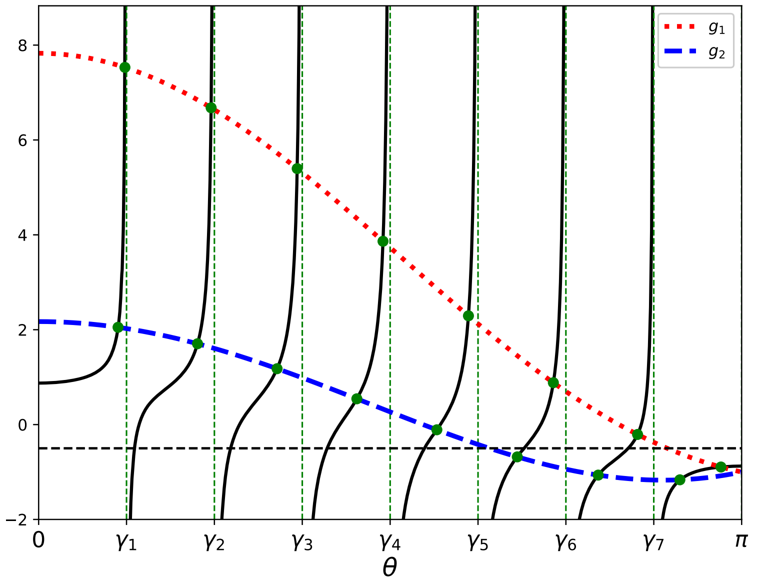

Fig. 5. Graph of the functions $g_1(\theta), g_2(\theta)$, and $F(\theta)$ (black) for $\theta \in [0, \pi]$ for $k = 8$.

The explicit expressions for g_1 and g_2 are

$$g_1(\theta) = 2 + 3\cos(\theta) + \sqrt{(\cos\theta + 1)(\cos\theta + 3)}$$

$$g_2(\theta) = 2 + 3\cos(\theta) - \sqrt{(\cos\theta + 1)(\cos\theta + 3)}.$$

The graphs of g_1, g_2, and F on the interval $[0, \pi]$ are shown in Fig. 5. In this case, the singularities of F occur at the equally spaced points

$$\gamma_j = \frac{j\pi}{k}, \quad j = 1, 2, \ldots, k.$$

If $\theta_1^+ < \theta_2^+ < \cdots < \theta_k^+$ denote the unique points where F and g_1 intersect then $f_1(\theta_1^+) < f_1(\theta_2^+) < \cdots < f_1(\theta_k^+)$ are the positive eigenvalues of A_{2k+1}. Similarly, if $\theta_1^- < \theta_2^- < \cdots < \theta_k^-$ denote the unique points where F and g_2 intersect then $f_2(\theta_k^-) < f_2(\theta_{k-1}^-) < \cdots < f_2(\theta_1^-)$ are the negative eigenvalues of A_{2k+1}.

Theorems 2.1–2.3 hold for the odd case with now $\lambda = 0$ being the trivial eigenvalue. Theorem 5.2, Theorem 6.1, and Theorem 6.2 proved for the even case hold almost verbatim for the odd case; the only change is that the ratio $\frac{2\pi}{2k-1}$ is now $\frac{\pi}{k}$.

8. The eigenvalues of threshold graphs

In this section, we discuss how a characterization of the eigenvalues of A_n could be used to characterize the eigenvalues of general threshold graphs. Let G be a threshold graph with binary creation sequence $b = (0^{s_1}, 1^{t_1}, \cdots, 0^{s_k}, 1^{t_k})$, where 0^{s_i} is shorthand for $s_i \geq 0$ consecutive zeros, and similarly for 1^{t_i}. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ denote the associated canonical labeling of G consistent with b. The set partition $\pi = \{C_1, C_2, \ldots, C_{2k}\}$ of $V(G)$ where C_1 contains the first s_1 vertices, C_2 contains the next t_1 vertices, and so on, is an equitable partition of G. The $2k \times 2k$ quotient graph G/π has adjacency matrix

$$A_\pi = A_{2k} + \operatorname{diag}(0, \beta_1, \ldots, 0, \beta_k)$$

where A_{2k} is the adjacency matrix of the connected anti-regular graph with $2k$ vertices and $\beta_i = 1 - \frac{1}{t_i}$, see for instance [13]. The eigenvalues of G other than the trivial eigenvalues $\lambda = -1$ and/or $\lambda = 0$ are exactly the eigenvalues of A_π. Presumably, the characterization of the eigenvalues of A_{2k} that we have done in this paper will be useful in characterizing the eigenvalues of A_π. We leave this investigation for a future paper.

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References

[1] R. Bapat, On the adjacency matrix of a threshold graph, Linear Algebra Appl. 439 (10) (2013) 3008–3015.
[2] M. Behzad, G. Chartrand, No graph is perfect, Amer. Math. Monthly 74 (8) (1967) 962–963.
[3] R. Merris, Antiregular graphs are universal for trees, Publ. Elektroteh. Fak. Ser. Mat. (2003) 1–3.
[4] N.V.R. Mahadev, U.N. Peled, Threshold Graphs and Related Topics, vol. 56, Elsevier, 1995.
[5] Martin C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, vol. 57, Elsevier, 2004.
[6] E. Munarini, Characteristic, admittance, and matching polynomials of an antiregular graph, Appl. Anal. Discrete Math. 3 (1) (2009) 157–176.
[7] V.E. Levit, E. Mandrescu, On the independence polynomial of an antiregular graph, Carpathian J. Math. 28 (2) (2012) 279–288.
[8] J.C. Mason, D.C. Handscomb, Chebyshev Polynomials, Chapman and Hall/CRC, 2002.
[9] D. Kulkarni, D. Schmidt, S.D. Tsui, Eigenvalues of tridiagonal pseudo-Toeplitz matrices, Linear Algebra Appl. 297 (1999) 63–80.
[10] D. Jacobs, V. Trevisan, F. Tura, Eigenvalues and energy in threshold graphs, Linear Algebra Appl. 465 (2015) 412–425.
[11] V. Chvátal, P.L. Hammer, Aggregation of inequalities in integer programming, Ann. Discrete Math. 1 (1977) 145–162.
[12] P.B. Henderson, Y. Zalcstein, A graph-theoretic characterization of the PV class of synchronizing primitives, SIAM J. Comput. 6 (1) (1977) 88–108.
[13] A. Banerjee, R. Mehatari, On the normalized spectrum of threshold graphs, Linear Algebra Appl. 530 (2017) 288–304.
[14] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.