

SMALL-TIME LOCAL CONTROLLABILITY FOR A CLASS OF HOMOGENEOUS SYSTEMS*

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Abstract. In this paper we consider the local controllability problem for control-affine systems that are homogeneous with respect to a one-parameter family of dilations corresponding to time-scaling in the control. We construct and derive properties of a variational cone that completely characterizes local controllability for these homogeneous systems. In the process, we are able to give a bound on the order, in terms of the integers describing the dilation, of perturbations that do not alter the local controllability property. Our approach uses elementary Taylor expansions and avoids unnecessarily complicated open mapping theorems to prove local controllability. Examples are given that illustrate the main results.

Key words. local controllability at a point, high-order variations, control-affine systems, homogeneous systems

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1. Introduction. The property of homogeneity is a key ingredient in many interesting results on local controllability and stabilizability of nonlinear control systems; see for instance [4, 12, 15] and references therein. In this paper, we consider the small-time local controllability of homogeneous control-affine systems

$$(1.1) \quad \Sigma : \dot{x}(t) = X_0(x) + \sum_{a=1}^m u_a X_a(x), \quad x(0) = x_0,$$

where X_0, X_1, \dots, X_m are smooth vector fields on a smooth manifold M with $X_0(x_0) = 0_{x_0}$, and the controls $t \mapsto u(t) = (u_1(t), \dots, u_m(t))$ are piecewise constant taking their values in a set $U \subset \mathbb{R}^m$, assumed to contain a neighborhood of the origin $0 \in \mathbb{R}^m$. We say that Σ is *small-time locally controllable* (STLC) from x_0 if the reachable set of Σ from x_0 in time at most $T > 0$, that is, the set

$$\mathcal{R}(x_0, T) = \bigcup_{0 \leq t \leq T} \{ \gamma(t) \mid \gamma : [0, t] \rightarrow M \text{ satisfies (1.1) for some control } u \}$$

contains x_0 in its interior for each $T > 0$. The concept of homogeneity that we employ rests on the notion of a *one-parameter family of dilations* [8], by which we mean a map $\Delta : \mathbb{R}_{>0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$(1.2) \quad \Delta(s, x_1, \dots, x_n) = (s^{k_1} x_1, s^{k_2} x_2, \dots, s^{k_n} x_n)$$

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for positive integers $k_1 \leq k_2 \leq \dots \leq k_n$. Throughout the paper, we denote $\Delta(s, \cdot)$ by Δ_s . Given a dilation Δ , we say that a control-affine system Σ on $M = \mathbb{R}^n$ is Δ -homogeneous if for every trajectory $\gamma : [0, T] \rightarrow \mathbb{R}^n$ of Σ , corresponding to the control $u : [0, T] \rightarrow U$, it holds that $\gamma_s(st) = \Delta_s(\gamma(t))$ for all $s > 0$, where $\gamma_s : [0, sT] \rightarrow \mathbb{R}^n$ is the trajectory of Σ corresponding to the scaled control $u_s : [0, sT] \rightarrow U$ defined as $u_s(st) = u(t)$. We note that we are only considering systems that are homogeneous with respect to time-scalings in the control and not a more general notion of homogeneity where the controls can also be scaled by their magnitudes, e.g., [20]. However, we remark that, even for this restricted class of homogeneous systems, sharp conditions for STLC are lacking. In this regard, one of the main contributions of our paper is a *necessary* condition for STLC for the type of homogeneous systems in consideration which, to the best of the authors knowledge, is missing in the literature.

The local controllability problem has a long and rich history. Since the late 1970s, much of the work on local controllability has been concerned with deriving Lie bracket conditions for establishing the STLC property or lack thereof. This effort can be explained by a result due to Nagano [18] relating diffeomorphism invariant properties, such as STLC, and Lie bracket relations of families of real analytic vector fields. Much of the work along these lines initiated with Hermes [10, 11] and was thoroughly developed by Sussmann [20] and Bianchini and Stefani [6]; many others have made significant contributions but our purpose is not to give an exhaustive survey. Although the current sufficient conditions as given in [20, 6] are rather general, they fail to capture the STLC property for relatively simple (polynomial) systems. For example, the control-affine system on $M = \mathbb{R}^4$ given by

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = \frac{1}{6}x_1^3, \quad \dot{x}_4 = x_2x_3$$

fails the well-known sufficient condition in [20, Theorem 7.3], yet STLC for this system can be proved using its homogeneity properties (see Example 5.2 and [14]). This example, and several others [14], demonstrate the gap between the known sufficient and necessary Lie bracket conditions for STLC. The purpose of this paper is not to narrow the gap by giving new Lie bracket conditions but instead to show that for the class of homogeneous systems in consideration, STLC can be completely characterized by a certain variational cone (Theorem 4.1) and that any control-affine system $\tilde{\Sigma}$, whose Taylor approximation up to order $k_n - 1$ at x_0 agrees with that of Σ , is STLC from x_0 if Σ is STLC from x_0 (Theorem 4.3). Although our results do not give explicit computational Lie bracket conditions, they identify a particularly simple type of variation to study STLC for an important class of homogeneous systems. Specifically, Theorem 4.1 gives a sufficient and necessary condition for STLC in terms of classical variations and potentially can be used as a guide to narrow the gap between the known conditions for STLC in terms of Lie brackets. Moreover, the proof of Theorem 4.1 gives an algorithmic procedure for determining STLC for the class of homogeneous systems considered when the known sufficient conditions fail. Our approach uses Taylor expansions of a composition of flows of vector fields as opposed to using the Campbell–Baker–Hausdorff formula or the more general formalism of chronological calculus [2]. Hence, a contribution of our paper is a self-contained and straightforward exposition of the characterization of STLC for an important class of nonlinear control-affine systems. In summary, the primary contributions of this work are

- a sufficient and necessary condition for STLC for control-affine systems that are homogeneous with respect to a family of dilations corresponding to time-scaling in the control (Theorem 4.1),

- a bound on the order of perturbations that do not alter the STLC property for control-affine systems that are homogeneous with respect to a family of dilations corresponding to time-scaling in the control (Theorem 4.3), and
- a self-contained development of the main results.

Our contributions are significant for two main reasons. First, aside from linear and driftless systems, the authors are unaware of any general result such as Theorem 4.1 that provides a sufficient and necessary condition for STLC in terms of variations or Lie brackets. Second, Theorem 4.3 establishes a bound on the order of derivatives needed to establish STLC for the class of homogeneous systems in consideration, and thus answers a question posed in [16] regarding the stability of STLC with respect to high-order perturbations.

This paper is organized as follows. In section 2 we construct a type of high-order tangent vector, or *variation*, using a composition of flows of vector fields and in section 3 use them to define a variational cone for control-affine systems. The use of variations to study the reachable set is of course not new, and the specific type of variations used here have been used at least as early by Krener [17] to prove the high-order maximum principle. The properties of these variations proved in section 3 parallel the development of the more general variations constructed in [17]. However, as these simpler variations suffice to characterize the STLC property for the systems we consider, we include all proofs and details to make this paper as self-contained as possible. Moreover, as will be shown in section 3, our constructions lead to the use of an elementary open mapping theorem to prove STLC, and furthermore, we are able to prove a theorem on subspaces of variations (Theorem 3.6) using our formalism. In section 4 we present our main results for the type of homogeneous systems considered, and finally in section 5 we illustrate our main theorems with some examples.

1.1. Notation and conventions. In this paper, vector fields will be used in both the geometric and algebraic sense. That is, a vector field ξ on a smooth manifold M will be thought of as a section of the tangent bundle TM and also as a derivation on the ring of smooth functions on M . In the latter case, the action of ξ on a smooth function $f : M \rightarrow \mathbb{R}$ will be denoted as ξf . Similarly, given a tangent vector $v \in T_x M$, the directional derivative of f with respect to v will be denoted by $v f$. Given two vector fields ξ and η , the product $\xi\eta$ will denote the differential operator $(\xi\eta)(f) = \xi(\eta f)$. We will use the shorthand notation ξ^2 to denote $\xi\xi$, ξ^3 to denote $\xi\xi\xi$, etc. The Lie bracket of ξ and η will be denoted by $[\xi, \eta]$. We also denote $\text{ad}_\xi^0 \eta = \eta$ and $\text{ad}_\xi^\ell \eta = [\xi, \text{ad}_\xi^{\ell-1} \eta]$ for $\ell \geq 1$.

We use the notation $\mathbb{R}_{\geq 0}^p = \{(\tau_1, \dots, \tau_p) \in \mathbb{R}^p : \tau_i \geq 0, i = 1, \dots, p\}$. Also, a control-affine system of the form (1.1) will be denoted by $\Sigma = (\{X_0, X_1, \dots, X_m\}, U)$.

2. Variations. For a smooth vector field ξ on M , its flow will be denoted by $(t, x) \mapsto \Phi^\xi(t, x) = \Phi_t^\xi(x) = \Phi_x^\xi(t)$, which is defined for all (t, x) in an open subset of $\mathbb{R} \times M$. More generally, if $\xi = (\xi_1, \dots, \xi_p)$ is a family of smooth vector fields on M , define the mapping $\Phi^\xi : \Omega_\xi \rightarrow M$ by

$$\Phi^\xi(t, x) = \Phi_{t_p}^{\xi_p} \circ \Phi_{t_{p-1}}^{\xi_{p-1}} \circ \dots \circ \Phi_{t_1}^{\xi_1}(x),$$

where $t = (t_1, \dots, t_p)$ and Ω_ξ is an open subset of $\mathbb{R}^p \times M$. For fixed $t \in \mathbb{R}^p$, we let Φ_t^ξ denote the map $x \mapsto \Phi_t^\xi(x) = \Phi^\xi(t, x)$ (when it exists), and for fixed $x \in M$, Φ_x^ξ is the map defined as $t \mapsto \Phi_x^\xi(t) = \Phi^\xi(t, x)$, which is defined in a neighborhood of the origin in \mathbb{R}^p . Henceforth, for ease of presentation we omit explicitly stating the

domain of definition of composition of flows of vector fields, understanding that they are defined only locally.

For a positive integer p let ET_p denote the set of smooth mappings $\tau : [0, 1] \rightarrow \mathbb{R}_{\geq 0}^p$ such that $\tau(0) = 0$. An element of ET_p will be called an *end-time*. Given a family of vector fields $\xi = (\xi_1, \dots, \xi_p)$, $\tau \in ET_p$, and $\epsilon > 0$ sufficiently small, the composite map $\Phi_{x_0}^\xi \circ \tau : [0, \epsilon] \rightarrow M$ is a well-defined curve at x_0 whose image consists of points obtained by following (in forward time) concatenations of the integral curves of ξ_1, \dots, ξ_p . The order of the pair (ξ, τ) at x_0 , denoted $\text{ord}_{x_0}(\xi, \tau)$, is the smallest integer $k \geq 1$ such that

$$\left. \frac{d^k}{ds^k} \right|_{s=0} \Phi_{x_0}^\xi(\tau(s)) \neq 0_{x_0}$$

provided such an integer exists, where $0_{x_0} \in T_{x_0}M$ denotes the zero tangent vector at x_0 . If $k = \text{ord}_{x_0}(\xi, \tau)$, we call

$$v_{\xi, \tau} := \left. \frac{d^k}{ds^k} \right|_{s=0} \Phi_{x_0}^\xi(\tau(s))$$

the (ξ, τ) -*end-time variation* or just *variation* when (ξ, τ) is understood.

To better understand how a variation $v_{\xi, \tau}$ depends on the jets of ξ at x_0 , by the chain rule, we need to compute the Taylor series of the maps $\Phi_{x_0}^\xi$ at the origin. To this end, we first introduce some standard multi-index notation. For a multi-index $I = (i_1, \dots, i_p)$, we let $|I| = i_1 + \dots + i_p$ and let $I! = i_1! \cdots i_p!$. For a family of vector fields $\xi = (\xi_1, \dots, \xi_p)$, a multi-index $I = (i_1, \dots, i_p)$, and a smooth function $f : M \rightarrow \mathbb{R}$, let $\xi^I f : M \rightarrow \mathbb{R}$ be the function defined by $(\xi^I f)(x) = (\xi_1^{i_1} \cdots \xi_p^{i_p} f)(x)$. For $t = (t_1, \dots, t_p) \in \mathbb{R}^p$ and a multi-index $I = (i_1, \dots, i_p)$, we set $t^I = t_1^{i_1} \cdots t_p^{i_p}$. The proof of the following is straightforward and will be omitted.

PROPOSITION 2.1. *Let $f : M \rightarrow \mathbb{R}$ be a smooth function, let $\xi = (\xi_1, \dots, \xi_p)$ be a family of smooth vector fields on M , and let $x_0 \in M$. The Taylor series at the origin of \mathbb{R}^p of the function $\mathbb{R}^p \ni t \mapsto (f \circ \Phi_{x_0}^\xi)(t)$ is*

$$\sum_{|I|=0}^{\infty} (\xi^I f)(x_0) \frac{t^I}{I!}.$$

Given a family of vector fields $\xi = (\xi_1, \dots, \xi_p)$, a smooth function $f : M \rightarrow \mathbb{R}$, and $x_0 \in M$, we denote by $(f \circ \Phi_{x_0}^\xi)_k$ the Taylor approximation of $f \circ \Phi_{x_0}^\xi$ of order $k \geq 1$. Explicitly,

$$(2.1) \quad (f \circ \Phi_{x_0}^\xi)_k(t) = \sum_{|I|=0}^k (\xi^I f)(x_0) \frac{t^I}{I!}.$$

It will be important for us to know how the Taylor polynomials (2.1) decompose when we view $\xi = (\xi_1, \dots, \xi_p)$ as being a concatenation of two families of vector fields. In what follows, given $\xi_1 = (\xi_{1,1}, \dots, \xi_{1,p})$ and $\xi_2 = (\xi_{2,1}, \dots, \xi_{2,q})$ we set $\xi_1 * \xi_2 = (\xi_{1,1}, \dots, \xi_{1,p}, \xi_{2,1}, \dots, \xi_{2,q})$.

LEMMA 2.2. *Let ξ_1 and ξ_2 be families of smooth vector fields on M of length p and q , respectively, and let $f : M \rightarrow \mathbb{R}$ be a smooth function that vanishes at x_0 . Let $\xi = \xi_1 * \xi_2$. Then, for each positive integer k and $(t_1, t_2) \in \mathbb{R}^p \times \mathbb{R}^q$,*

$$(f \circ \Phi_{x_0}^{\xi_1 * \xi_2})_k(t_1, t_2) = (f \circ \Phi_{x_0}^{\xi_1})_k(t_1) + (f \circ \Phi_{x_0}^{\xi_2})_k(t_2) + R_k^\xi(t_1, t_2),$$

where the remainder term is

$$R_k^\xi(\mathbf{t}_1, \mathbf{t}_2) = \sum_{|J|=1}^{k-1} \frac{\mathbf{t}_2^J}{J!} (h_J \circ \Phi_{x_0}^{\xi_1})_{k-|J|}(\mathbf{t}_1) \quad \text{and} \quad h_J = \xi_2^J f - \xi_2^J f(x_0).$$

Proof. From (2.1),

$$(f \circ \Phi_{x_0}^{\xi_1 * \xi_2})_k(\mathbf{t}_1, \mathbf{t}_2) = (f \circ \Phi_{x_0}^{\xi_1})_k(\mathbf{t}_1) + (f \circ \Phi_{x_0}^{\xi_2})_k(\mathbf{t}_2) + \sum_{\substack{|I|+|J|=2 \\ |I|, |J| \geq 1}}^k (\xi_1^I \xi_2^J f)(x_0) \frac{\mathbf{t}_1^I \mathbf{t}_2^J}{I! J!}.$$

Now, directly,

$$\begin{aligned} \sum_{\substack{|I|+|J|=2 \\ |I|, |J| \geq 1}}^k (\xi_1^I \xi_2^J f)(x_0) \frac{\mathbf{t}_1^I \mathbf{t}_2^J}{I! J!} &= \sum_{|J|=1}^{k-1} \sum_{|I|=1}^{k-|J|} (\xi_1^I \xi_2^J f)(x_0) \frac{\mathbf{t}_1^I \mathbf{t}_2^J}{I! J!} \\ &= \sum_{|J|=1}^{k-1} \frac{\mathbf{t}_2^J}{J!} \sum_{|I|=1}^{k-|J|} \xi_1^I (\xi_2^J f - \xi_2^J f(x_0))(x_0) \frac{\mathbf{t}_1^I}{I!} \\ &= \sum_{|J|=1}^{k-1} \frac{\mathbf{t}_2^J}{J!} (h_J \circ \Phi_{x_0}^{\xi_1})_{k-|J|}(\mathbf{t}_1), \end{aligned}$$

where the last equality follows because the function $x \mapsto h_J(x) = \xi_2^J f(x) - \xi_2^J f(x_0)$ vanishes at x_0 . \square

LEMMA 2.3. *Let ξ be a family of smooth vector fields of length p and let $\tau \in \text{ET}_p$. Suppose that $k = \text{ord}_{x_0}(\xi, \tau) \geq 2$ and let $\rho : \mathbb{R} \rightarrow \mathbb{R}^q$ be a smooth map such that $\rho(0) = 0$. For any smooth function $f : M \rightarrow \mathbb{R}$ and any multi-index $J = (j_1, \dots, j_q)$ with $1 \leq |J| \leq k - 1$, the derivatives of the function $s \mapsto \rho^J(s) (f \circ \Phi_{x_0}^\xi)_{k-|J|}(\tau(s))$ of orders $0, 1, \dots, k$ vanish at $s = 0$, where we denote $\rho^J(s) = (\rho_1(s))^{j_1} \cdots (\rho_q(s))^{j_q}$.*

Proof. Suppose that $1 \leq |J| \leq k - 1$. By the Leibniz rule, the derivatives of the function $s \mapsto \rho^J(s)$ of orders $0, 1, \dots, |J| - 1$ all vanish at $s = 0$. By definition of ord_{x_0} , the derivatives of the function $s \mapsto (f \circ \Phi_{x_0}^\xi)_{k-|J|}(\tau(s))$ of orders $1, \dots, k - |J|$ all vanish at $s = 0$. Therefore, by the Leibniz rule, the derivatives of the function $s \mapsto \rho^J(s) (f \circ \Phi_{x_0}^\xi)_{k-|J|}(\tau(s))$ of orders $0, 1, \dots, k$ all vanish at $s = 0$. \square

3. A variational cone. In this section we fix a control-affine system Σ and define the family of vector fields $\mathcal{F}_\Sigma = \{X_0 + \sum_{a=1}^m u_a X_a : u \in U\}$. Let \mathcal{F}_Σ^p denote the set of p -tuples of elements of \mathcal{F}_Σ . For a positive integer k let

$$\mathcal{V}_{x_0}^k = \{v_{\xi, \tau} : (\xi, \tau) \in \cup_{p \geq 1} (\mathcal{F}_\Sigma^p \times \text{ET}_p), \text{ord}_{x_0}(\xi, \tau) = k\} \cup \{0_{x_0}\}$$

and let

$$\mathcal{V}_{x_0} = \bigcup_{k \geq 1} \mathcal{V}_{x_0}^k.$$

By definition, \mathcal{V}_{x_0} is a set of high-order tangent vectors at x_0 to the reachable set of Σ from x_0 . In this section, we will show that \mathcal{V}_{x_0} is an approximating cone to the reachable set of Σ in the sense that if $\mathcal{V}_{x_0} = T_{x_0}M$, then Σ is STLC from x_0 . More general notions of variations can be found in, for example, [17, 7, 14, 5] with their corresponding approximating theorems. To keep this paper as self-contained as possible, however, we include all proofs as they involve only elementary Taylor series computations and a degree theory argument (Lemma 3.5).

To prove the main property of \mathcal{V}_{x_0} that allows it to serve as an approximation to $\mathcal{R}(x_0, T)$, we first note that a curve $c : \mathbb{R} \rightarrow M$ is of order k at 0 if and only if for any smooth function $f : M \rightarrow \mathbb{R}$, the derivatives at 0 of the function $f \circ c$ vanish up to order $k - 1$, and in this case

$$\left. \frac{d^k}{ds^k} \right|_{s=0} f(c(s)) = v f,$$

where $v = c^{(k)}(0) \in T_{c(0)}M$. Therefore, if $k = \text{ord}_{x_0}(\xi, \tau)$, then for any smooth function $f : M \rightarrow \mathbb{R}$, the derivatives of the function $(f \circ \Phi_{x_0}^\xi)_k \circ \tau : [0, \epsilon] \rightarrow \mathbb{R}$ vanish up to order $k - 1$ at 0, and

$$\left. \frac{d^k}{ds^k} \right|_{s=0} (f \circ \Phi_{x_0}^\xi)_k(\tau(s)) = v_{\xi, \tau} f.$$

PROPOSITION 3.1. *The set $\mathcal{V}_{x_0}^k$ is a convex cone.*

Proof. We first prove that $\mathcal{V}_{x_0}^k$ is closed under addition. Let $(\xi_1, \tau_1), (\xi_2, \tau_2)$ be of order k at x_0 , set $\xi = \xi_1 * \xi_2$, and set $\tau = \tau_1 * \tau_2$. We claim that (ξ, τ) is of order k at x_0 and that $v_{\xi, \tau} = v_{\xi_1, \tau_1} + v_{\xi_2, \tau_2}$. To prove this, we can assume that $v_{\xi_1, \tau_1} \neq -v_{\xi_2, \tau_2}$; if not, then $v_{\xi_1, \tau_1} + v_{\xi_2, \tau_2} = 0_{x_0} \in \mathcal{V}_{x_0}^k$. Let $f : M \rightarrow \mathbb{R}$ be a smooth function that vanishes at x_0 . By Lemma 2.2,

$$(3.1) \quad (f \circ \Phi_{x_0}^\xi)_k(\tau(s)) = (f \circ \Phi_{x_0}^{\xi_1})_k(\tau_1(s)) + (f \circ \Phi_{x_0}^{\xi_2})_k(\tau_2(s)) + R_k^\xi(\tau_1(s), \tau_2(s)),$$

where

$$R_k^\xi(\tau_1(s), \tau_2(s)) = \sum_{|J|=1}^{k-1} \frac{\tau_2^J(s)}{J!} (h_J \circ \Phi_{x_0}^{\xi_1})_{k-|J|}(\tau_1(s)),$$

and $h_J = \xi_2^J f - \xi_2^J f(x_0)$. By Lemma 2.3, the first k derivatives of the function $s \mapsto R_k^\xi(\tau_1(s), \tau_2(s))$ vanish at $s = 0$. Therefore, $k = \text{ord}_{x_0}(\xi, \tau)$ and from (3.1) we have

$$\left. \frac{d^k}{ds^k} \right|_{s=0} (f \circ \Phi_{x_0}^\xi)_k(\tau(s)) = v_{\xi_1, \tau_1} f + v_{\xi_2, \tau_2} f = (v_{\xi_1, \tau_1} + v_{\xi_2, \tau_2}) f,$$

which proves the claim.

To prove that $\mathcal{V}_{x_0}^k$ is closed under $\mathbb{R}_{>0}$ -multiplication, suppose that (ξ, τ) is of order k at x_0 , let $\alpha \in \mathbb{R}_{>0}$, and define τ_α by $\tau_\alpha(s) = \tau(\alpha^{1/k} s)$. By the chain rule, for all $\ell \in \mathbb{Z}_{>0}$,

$$\left. \frac{d^\ell}{ds^\ell} \right|_{s=0} \Phi_{x_0}^\xi(\tau_\alpha(s)) = \alpha^{\ell/k} \left. \frac{d^\ell}{ds^\ell} \right|_{s=0} \Phi_{x_0}^\xi(\tau(s)).$$

Therefore, (ξ, τ_α) is of order k at x_0 and $v_{\xi, \tau_\alpha} = \alpha v_{\xi, \tau}$. This completes the proof. \square

The next key property that is needed to use \mathcal{V}_{x_0} as an approximation to $\mathcal{R}(x_0, T)$ is a nesting type condition.

LEMMA 3.2 (see [17]). *For positive integers k and m , $\mathcal{V}_{x_0}^k \subseteq \mathcal{V}_{x_0}^{km}$.*

Proof. If (ξ, τ) is of order k at x_0 , then, for any function f vanishing at x_0 ,

$$(f \circ \Phi_{x_0}^\xi)_k(\tau(s)) = (v_{\xi, \tau} f) \frac{s^k}{k!} + o(s^k).$$

Therefore,

$$(f \circ \Phi_{x_0}^{\xi})_k(\tau((k!/(km)!)^{1/k} s^m)) = (v_{\xi, \tau} f) \frac{s^{km}}{(km)!} + o(s^{km}).$$

It follows that if

$$\rho(s) = \tau((k!/(km)!)^{1/k} s^m),$$

then (ξ, ρ) is of order km at x_0 and $v_{\xi, \rho} = v_{\xi, \tau}$. \square

COROLLARY 3.3. \mathcal{V}_{x_0} is a convex cone.

Proof. The set \mathcal{V}_{x_0} is a cone because it is a union of cones. By Lemma 3.2, if $v_1, \dots, v_r \in \mathcal{V}_{x_0}$, with $v_j \in \mathcal{V}_{x_0}^{k_j}$ and $k = \text{lcm}(k_1, \dots, k_r)$, then $v_1, \dots, v_r \in \mathcal{V}_{x_0}^k$. By Proposition 3.1, $\mathcal{V}_{x_0}^k$ is convex and, therefore, any convex combination of v_1, \dots, v_r is an element of $\mathcal{V}_{x_0}^k \subset \mathcal{V}_{x_0}$. This completes the proof. \square

Remark 3.1. Our definition of a variation uses smooth functions $\tau : [0, 1] \rightarrow \mathbb{R}_{\geq 0}^p$, so that in general we do not have $\mathcal{V}_{x_0}^k \subseteq \mathcal{V}_{x_0}^{k+1}$. If the end-times τ are allowed to be C^r at $s = 0$ for $r \geq 1$, then a variation of order k can be realized as a variation of order $\ell > k$ after a reparameterization. However, one then needs to keep track of the order of differentiability of the end-times τ to be able to work with high-order jets. For this reason we choose to work with smooth end-times, and Lemma 3.2 ensures that essentially nothing is lost by doing so. The use of smooth end-times are employed, for instance, in [17], whereas [11] uses end-times that are C^r , $r \geq 1$.

The following theorem relates \mathcal{V}_{x_0} and STLC of Σ at x_0 . To prove the theorem, one can use the general results of [7, 5, 14]. By contrast, our proof relies on the algebraic properties of \mathcal{V}_{x_0} proven thus far and on a relatively simple open mapping theorem (Lemma 3.5 below).

THEOREM 3.4. *Let Σ be a control-affine system of the form (1.1). If $\mathcal{V}_{x_0} = T_{x_0}M$, then Σ is STLC from x_0 .*

Proof. Let $T > 0$ be given. By assumption, there exists $v_{\xi_1, \tau_1}, \dots, v_{\xi_r, \tau_r} \in \mathcal{V}_{x_0}$ such that

$$(3.2) \quad 0 \in \text{int}(\text{co}(\{v_{\xi_1, \tau_1}, \dots, v_{\xi_r, \tau_r}\})).$$

In (3.2), $\text{co}(\cdot)$ and $\text{int}(\cdot)$ denote the convex hull and interior, respectively. By Lemma 3.2, we can assume that $v_{\xi_i, \tau_i} \in \mathcal{V}_{x_0}^k$ for some $k \in \mathbb{Z}_{>0}$ for all $i = 1, \dots, r$. Consider the map $\mu : \Omega \cap \mathbb{R}_{\geq 0}^r \rightarrow M$ defined by

$$\mu(s_1, \dots, s_r) = \Phi_{\tau_1((k!s_1)^{1/k})}^{\xi_1} \circ \dots \circ \Phi_{\tau_r((k!s_r)^{1/k})}^{\xi_r}(x_0),$$

where Ω is a neighborhood of the origin in \mathbb{R}^r with the property that if $(s_1, \dots, s_r) \in \Omega \cap \mathbb{R}_{\geq 0}^r$, then $\sum_{i,j} \tau_{j,i}((k!s_j)^{1/k}) \leq T$. By construction, μ is differentiable at the origin, $\mu(0) = x_0$, and the image of μ consists of points reachable from x_0 in time at most T . It is clear that $\frac{\partial \mu}{\partial s_i}(0) = v_{\xi_i, \tau_i}$ for $i = 1, \dots, r$, and therefore $D\mu(0)(\mathbb{R}_{\geq 0}^r) = T_{x_0}M$ by (3.2). Applying Lemma 3.5 below to (the coordinate representation of) μ then implies that $x_0 \in \text{int}(\mathcal{R}(x_0, T))$. This completes the proof. \square

LEMMA 3.5 (see [3]). *Let $\mu : \mathbb{R}^r \rightarrow \mathbb{R}^n$ be Lipschitzian, $\mu(0) = 0$, and differentiable at 0. Assume that $D\mu(0)(\mathbb{R}_{\geq 0}^r) = \mathbb{R}^n$. Then $0 \in \text{int}(\mu(\Omega \cap \mathbb{R}_{\geq 0}^r))$ for any neighborhood Ω of the origin in \mathbb{R}^r .*

3.1. Subspaces of variations. Before moving on to homogeneous systems, in this section we construct linear approximations to the convex cone \mathcal{V}_{x_0} . Explicitly, using a technique from Krener [17, section 4], we construct subspaces of variations. The main result of this section (Theorem 3.6) implies that

$$\text{span}\{\text{ad}_{X_0}^{\ell_1}(X_j)(x_0), \text{ad}_{X_0}^{\ell_2}([X_i, X_j])(x_0) \mid \ell_1, \ell_2 \geq 0, i, j = 1, \dots, m\}$$

is a subspace of variations, a result obtained in [6, Corollary 3.7] using a more general notion of a variation.

If ζ is a vector field on M that vanishes at x_0 , then ζ induces a canonical linear map $B_\zeta : T_{x_0}M \rightarrow T_{x_0}M$ defined by $B_\zeta(v) = [V, \zeta](x_0)$, where V is any vector field extending $v \in T_{x_0}M$. For a control-affine system Σ define

$$\mathcal{L}_{x_0} = \{\zeta \in \mathcal{F}_\Sigma : \zeta(x_0) = 0_{x_0}\}.$$

We identify \mathcal{L}_{x_0} with the corresponding subset of linear maps on $T_{x_0}M$, which we still denote by \mathcal{L}_{x_0} . For a subspace $W \subseteq T_{x_0}M$, let $\langle \mathcal{L}_{x_0}; W \rangle$ denote the smallest subspace containing W that is invariant under the linear maps in \mathcal{L}_{x_0} . It is not hard to show that

$$\langle \mathcal{L}_{x_0}; W \rangle = \text{span}\{B_{\zeta_1}B_{\zeta_2} \cdots B_{\zeta_r}(w) \mid w \in W, \zeta_i \in \mathcal{L}_{x_0}, r \in \mathbb{Z}_{\geq 0}\}.$$

THEOREM 3.6. *Let Σ be a smooth control-affine system and let $x_0 \in M$. For any subspace $W \subseteq \mathcal{V}_{x_0}$, it holds that $\langle \mathcal{L}_{x_0}; W \rangle \subseteq \mathcal{V}_{x_0}$.*

Proof. To prove the theorem, it is enough to show that, if $w \in W$ and $\zeta \in \mathcal{L}_{x_0}$, then $B_\zeta(w) \in \mathcal{V}_{x_0}$.

Let $w \in W$ and let $\zeta \in \mathcal{L}_{x_0}$. By Lemma 3.2, we can assume that there exists an integer $k \geq 1$ and (ξ_i, τ_i) of order k at x_0 such that $v_{\xi_i, \tau_i} = (-1)^{i+1}w$ for $i = 1, 2$. Let $\tilde{\tau}_i(s) = \tau_i((k!/(2k!)^{1/k}s^2)$ for $i = 1, 2$. Then, by the proof of Lemma 3.2, $\text{ord}_{x_0}(\xi_i, \tilde{\tau}_i) = 2k$ and $v_{\xi_i, \tilde{\tau}_i} = (-1)^{i+1}w$ for $i = 1, 2$. Now, since $\zeta(x_0) = 0_{x_0}$ and $v_{\xi_1, \tilde{\tau}_1} = -v_{\xi_2, \tilde{\tau}_2}$, we have that $\text{ord}_{x_0}(\xi_1 * \zeta * \xi_2, \tilde{\tau}_1 * s * \tilde{\tau}_2) \geq 2k + 1$. By definition and then expanding,

$$\begin{aligned} (3.3) \quad & (f \circ \Phi_{x_0}^{\xi_1 * \zeta * \xi_2})_{2k+1}(\tilde{\tau}_1(s), s, \tilde{\tau}_2(s)) \\ &= (f \circ \Phi_{x_0}^{\xi_1})_{2k+1}(\tilde{\tau}_1(s)) + (f \circ \Phi_{x_0}^{\xi_2})_{2k+1}(\tilde{\tau}_2(s)) \\ &+ (f \circ \Phi_{x_0}^\zeta)_{2k+1}(s) + \sum_{\substack{|I_1|+j=2 \\ |I_1|, j \geq 1}}^{2k+1} (\xi_1^{I_1} \zeta^j f)(x_0) \frac{s^j \tilde{\tau}_1^{I_1}(s)}{j! I_1!} \\ &+ \sum_{\substack{|I_2|+j=2 \\ |I_2|, j \geq 1}}^{2k+1} (\zeta^j \xi_2^{I_2} f)(x_0) \frac{s^j \tilde{\tau}_2^{I_2}(s)}{j! I_2!} \\ &+ \sum_{\substack{|I_1|+|I_2|=2 \\ |I_1|, |I_2| \geq 1}}^{2k+1} (\xi_1^{I_1} \xi_2^{I_2} f)(x_0) \frac{\tilde{\tau}_1^{I_1}(s) \tilde{\tau}_2^{I_2}(s)}{I_1! I_2!} \\ &+ \sum_{\substack{|I_1|+j+|I_2|=3 \\ |I_1|, j, |I_2| \geq 1}}^{2k+1} (\xi_1^{I_1} \zeta^j \xi_2^{I_2} f)(x_0) \frac{\tilde{\tau}_1^{I_1} s^j \tilde{\tau}_2^{I_2}}{I_1! j! I_2!}. \end{aligned}$$

Using the fact that $\zeta(x_0) = 0_{x_0}$ and letting h_j for each $j \in \{1, \dots, 2k\}$ be the smooth function $x \mapsto h_j(x) = (\zeta^j f)(x) - (\zeta^j f)(x_0)$, we can rewrite (3.3) as

$$(3.4) \quad (f \circ \Phi_{x_0}^{\xi_1 * \zeta * \xi_2})_{2k+1}(\tilde{\tau}_1(s), s, \tilde{\tau}_2(s)) = (f \circ \Phi_{x_0}^{\xi_1 * \xi_2})_{2k+1}(\tilde{\tau}_1(s), \tilde{\tau}_2(s)) \\ + \sum_{j=1}^{2k} \frac{s^j}{j!} (h_j \circ \Phi_{x_0}^{\xi_1})_{(2k+1)-j}(\tilde{\tau}_1(s)) \\ + \sum_{\substack{|I_1|+j+|I_2|=3 \\ |I_1|, j, |I_2| \geq 1}}^{2k+1} (\xi_1^{I_1} \zeta^j \xi_2^{I_2} f)(x_0) \frac{\tilde{\tau}_1^{I_1} s^j \tilde{\tau}_2^{I_2}}{I_1! j! I_2!}.$$

Now, $\text{ord}_{x_0}(\xi_1 * \xi_2, \tau_1 * \tau_2) \geq k + 1$ because $v_{\xi_1, \tau_1} + v_{\xi_2, \tau_2} = w - w = 0_{x_0}$, and therefore $\text{ord}_{x_0}(\xi_1 * \xi_2, \tilde{\tau}_1 * \tilde{\tau}_2) \geq 2(k + 1) = 2k + 2$. Hence, the derivatives of $(f \circ \Phi_{x_0}^{\xi_1 * \xi_2})_{2k+1}(\tilde{\tau}_1(s), \tilde{\tau}_2(s))$ of orders $1, \dots, 2k+1$ all vanish at $s = 0$. By Lemma 2.2, the last term in (3.5) can be written as

$$(3.5) \quad \sum_{j+|I_2|=2}^{2k} \frac{s^j \tilde{\tau}_2^{I_2}(s)}{j! I_2!} (H_{j, I_2} \circ \Phi_{x_0}^{\xi_1})_{(2k+1)-(j+|I_2|)}(\tilde{\tau}_1(s)),$$

where H_{j, I_2} is the smooth function $H_{j, I_2} = (\zeta^j \xi_2^{I_2} f) - (\zeta^j \xi_2^{I_2} f)(x_0)$. By Lemma 2.3, the derivatives of (3.5) up to order $2k + 1$ vanish at $s = 0$. Hence, $v_{\xi_1 * \zeta * \xi_2, \tilde{\tau}_1 * s * \tilde{\tau}_2}$ is determined by the $2k + 1$ derivative of the \mathbb{R} -valued function

$$s \mapsto g(s) := \sum_{j=1}^{2k} \frac{s^j}{j!} f_j(s),$$

where for each $j \in \{1, \dots, 2k\}$

$$f_j(s) = (h_j \circ \Phi_{x_0}^{\xi_1})_{(2k+1)-j}(\tilde{\tau}_1(s)).$$

Now since $\text{ord}_{x_0}(\xi_1, \tilde{\tau}_1) = 2k$, if $j \in \{2, \dots, 2k\}$, then the derivatives of f_j at $s = 0$ up to order $(2k + 1 - j)$ vanish. Therefore, the derivatives at $s = 0$ up to order $2k + 1$ of the function $s \mapsto s^j f_j(s)$ vanish for all $j \in \{2, \dots, 2k\}$. Thus the $2k + 1$ derivative at $s = 0$ of the function g is equal to the $2k + 1$ derivative at $s = 0$ of the function $s \mapsto s f_1(s)$. But the $2k$ th derivative of f_1 at $s = 0$ is

$$\left. \frac{d^{2k}}{ds^{2k}} \right|_{s=0} (h_1 \circ \Phi_{x_0}^{\xi_1})_{2k}(\tilde{\tau}_1(s)) = v_{\xi_1, \tilde{\tau}_1}(h_1) = w(\zeta f - \zeta f(x_0)) = B_\zeta(w)(f).$$

Hence, the $2k + 1$ derivative of $s \mapsto s f_1(s)$ is $(2k + 1)B_\zeta(w)(f)$. Therefore, we have $(2k + 1)B_\zeta(w) \in \mathcal{V}_{x_0}$, and since \mathcal{V}_{x_0} is a cone, $B_\zeta(w) \in \mathcal{V}_{x_0}$. This completes the proof. \square

Let us give an example of the previous theorem.

Example 3.1. On $M = \mathbb{R}^n$, let Σ be the linear control system $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and u lies in the unit cube in \mathbb{R}^m . Making the usual identifications on \mathbb{R}^n , it is clear that $\mathcal{V}_{x_0}^1 = \text{span}\{b_1, \dots, b_m\}$, where b_i is the i th column of B . The set \mathcal{X}_{x_0} contains the vector field $x \mapsto Ax$. Hence, by Theorem 3.6, the smallest subspace containing $\text{span}\{b_1, \dots, b_m\}$ and invariant under the linear vector field $x \mapsto Ax$ is a subspace of variations. In other words, the image of the classical Kalman controllability matrix $[B \ AB \ \dots \ A^{n-1}B]$ is a subspace of variations.

Remark 3.2. Theorem 3.6 is proved in [17, section 4] for the case of a single-input control-affine system.

4. Homogeneous systems. Homogeneous systems have received much attention in the literature with regards to controllability and stabilizability; see [12] for a survey. One of the basic problems is concerned with constructing homogeneous approximations that preserve the property of interest, for example, STLC or stabilizability. Our aim in this section is to show that, for a class of homogeneous systems, one can characterize the local controllability property with the variational cone constructed in section 3. In this section, $M = \mathbb{R}^n$.

We recall the definition of Δ -homogeneity from section 1. Given a control-affine system

$$(4.1) \quad \Sigma : \dot{x}(t) = X_0(x) + \sum_{a=1}^m u_a X_a(x), \quad x(0) = x_0,$$

we will say that (γ, u) is a *controlled trajectory of Σ on $[0, T]$* if $\gamma : [0, T] \rightarrow \mathbb{R}^n$ is the solution of (4.1) corresponding to the control $u : [0, T] \rightarrow U$. The set of controlled trajectories of Σ on $[0, T]$ will be denoted by $\text{Traj}_\Sigma(T)$. Given $(\gamma, u) \in \text{Traj}_\Sigma(T)$ and $s > 0$, define $(\gamma_s, u_s) \in \text{Traj}_\Sigma(sT)$ by setting $u_s(st) = u(t)$ for all $t \in [0, T]$. Given a one-parameter family of dilations $\{\Delta_s\}_{s>0}$ on \mathbb{R}^n , we say that Σ is Δ -homogeneous if for every $(\gamma, u) \in \text{Traj}_\Sigma(T)$ inducing (γ_s, u_s) it holds that $\gamma_s(st) = \Delta_s(\gamma(t))$ for all $t \in [0, T]$ and $s > 0$. A Δ -homogeneous system has, naturally, homogeneous reachable sets, that is, for each $T > 0$ and $s > 0$,

$$\mathcal{R}(x_0, sT) = \Delta_s(\mathcal{R}(x_0, T)).$$

This, for instance, implies that if $x_0 \in \text{int}(\mathcal{R}(x_0, t))$ for some $t > 0$, then $x_0 \in \text{int}(\mathcal{R}(x_0, T))$ for all $T > 0$.

Remark 4.1. The definition of homogeneity that we employ is equivalent to the notion of geometric or flow homogeneity as developed in [13, 15]. Following [15], let Z be a complete vector field on \mathbb{R}^n such that $-Z$ has $x_0 = 0$ as a global attractor. A vector field X is said to be Z -homogeneous of degree $\kappa \in \mathbb{Z}$ if

$$\Phi_s^Z \circ \Phi_t^X = \Phi_t^X \Phi_{e^{\kappa t} s}^Z.$$

It is straightforward to verify that X is Z -homogeneous if and only if $[Z, X] = \kappa X$. To relate the notion of Δ -homogeneity with Z -homogeneity, we say that a control-affine system $\Sigma = (\{X_0, X_1, \dots, X_m\}, U)$ is Z -homogeneous of degree κ if each X_i , $i = 0, 1, \dots, m$, is Z -homogeneous of degree κ . It is then straightforward to show that our definition for Σ to be Δ -homogeneous with respect to $\Delta(s, x) = (s^{k_1} x_1, \dots, s^{k_n} x_n)$ is equivalent to Σ being Z -homogeneous with $Z(x) = (k_1 x_1, \dots, k_n x_n)$. We remark that, as stated in the introduction, our notion of homogeneity does not include magnitude scalings of the control. In terms of geometric homogeneity as just defined, allowing magnitude scalings of the control translates to the possibility of having different degrees $\kappa_0, \kappa_1, \dots, \kappa_m$ of geometric homogeneity for the system vector fields X_0, X_1, \dots, X_m , respectively, with respect to Z .

Let us now state and prove the main result of this paper.

THEOREM 4.1. *Let Σ be a control-affine system on \mathbb{R}^n that is Δ -homogeneous with respect to the dilation $\Delta_s(x) = (s^{k_1} x_1, \dots, s^{k_n} x_n)$. Then Σ is STLC from $x_0 = 0$ if and only if*

$$\mathcal{V}_{x_0}^{k_1} + \mathcal{V}_{x_0}^{k_2} + \dots + \mathcal{V}_{x_0}^{k_n} = T_{x_0} \mathbb{R}^n.$$

Proof. If $\mathcal{V}_{x_0}^{k_1} + \mathcal{V}_{x_0}^{k_2} + \dots + \mathcal{V}_{x_0}^{k_n} = T_{x_0}\mathbb{R}^n$, then by Corollary 3.3 and Theorem 3.4 it follows that Σ is STLC from x_0 . Conversely, suppose that Σ is STLC from x_0 and let $T > 0$ be arbitrary. Let $\{e_1, \dots, e_n\}$ be the standard basis in \mathbb{R}^n and let $e_j \in \{e_1, \dots, e_n\}$ be arbitrary. By hypothesis, there is a controlled trajectory (γ, u) on $[0, T]$ and a constant $c > 0$ such that $\gamma(T) = ce_j$. In other words, there exists a family of vector fields $\xi = (\xi_1, \dots, \xi_p) \subset \mathcal{F}_\Sigma$, times $t_1, \dots, t_p > 0$ satisfying $t_1 + \dots + t_p = T$, such that

$$\gamma(T) = ce_j = \Phi_{t_p}^{\xi_p} \circ \dots \circ \Phi_{t_1}^{\xi_1}(x_0).$$

Consider the curve $\nu : [0, 1] \rightarrow \mathbb{R}^n$ given by

$$\nu(s) = \Phi_{t_p s}^{\xi_p} \circ \dots \circ \Phi_{t_1 s}^{\xi_1}(x_0).$$

By construction of ν , for $s \in (0, 1]$ it holds that $\nu(s) = \gamma_s(sT)$, where $(\gamma_s, u_s) \in \text{Traj}_\Sigma(sT)$ is induced by $(\gamma, u) \in \text{Traj}_\Sigma(T)$. By Δ -homogeneity and the fact that $\nu(0) = x_0$, it follows that $\nu(s) = ce_j s^{k_j}$ for all $s \in [0, 1]$. By construction of ν and the fact that $\mathcal{V}_{x_0}^{k_j}$ is a cone, it is clear that $\frac{\partial}{\partial x_j} \in \mathcal{V}_{x_0}^{k_j}$. An identical procedure shows that also $-\frac{\partial}{\partial x_j} \in \mathcal{V}_{x_0}^{k_j}$. This proves that $\mathcal{V}_{x_0}^{k_1} + \mathcal{V}_{x_0}^{k_2} + \dots + \mathcal{V}_{x_0}^{k_n} = T_{x_0}\mathbb{R}^n$. \square

By Lemma 3.2, the following corollary is immediate.

COROLLARY 4.2. *Let Σ be a control-affine system on \mathbb{R}^n that is Δ -homogeneous with respect to the dilation $\Delta_s(x) = (s^{k_1}x_1, \dots, s^{k_n}x_n)$. Let $k = \text{lcm}(k_1, \dots, k_n)$. Then Σ is STLC from $x_0 = 0$ if and only if $\mathcal{V}_{x_0}^k = T_{x_0}\mathbb{R}^n$.*

Remark 4.2. The *if* part of Theorem 4.1 still holds in the case of Lebesgue measurable controls, provided that we assume that the family \mathcal{F}_Σ satisfies the Lie algebra rank condition (LARC) at x_0 . Indeed, if the family \mathcal{F}_Σ satisfies the LARC at x_0 and Σ is STLC using Lebesgue measurable controls, then by a theorem of Grasse [9, Corollary 4.15], Σ is STLC using piecewise constant controls.

4.1. STLC preserved by high-order perturbations. In [16] (see also [1]), the following problem was posed. Suppose that the smooth control-affine system $\Sigma = (\{X_0, X_1, \dots, X_m\}, U)$ is STLC from x_0 . Does there exist an integer k such that every smooth control-affine system $\tilde{\Sigma} = (\{Y_0, Y_1, \dots, Y_m\}, U)$ is also STLC from x_0 if the Taylor expansions at x_0 of the vector fields of the two systems agree up to order k ? This problem remains open in the general case. For the class of homogeneous systems considered, Theorem 4.1 can be used to give a bound on the order of perturbations that do not alter STLC. In the following theorem, we will emphasize the dependence of $\mathcal{V}_{x_0}^k$ on Σ by writing of $\mathcal{V}_{\Sigma, x_0}^k$.

THEOREM 4.3. *Suppose that $\Sigma = (\{X_0, X_1, \dots, X_m\}, U)$ is Δ -homogeneous with respect to the dilation $\Delta_s(x) = (s^{k_1}x_1, \dots, s^{k_n}x_n)$. Let $\tilde{\Sigma} = (\{Y_0, Y_1, \dots, Y_m\}, U)$ be a control-affine system such that the Taylor expansion at x_0 of Y_i up to order $k_n - 1$ is equal to that of X_i for all $i = 0, 1, \dots, m$. If Σ is STLC from $x_0 = 0$, then so is $\tilde{\Sigma}$.*

Proof. If Σ is STLC from x_0 , by Theorem 4.1, $\mathcal{V}_{\Sigma, x_0}^{k_1} + \mathcal{V}_{\Sigma, x_0}^{k_2} + \dots + \mathcal{V}_{\Sigma, x_0}^{k_n} = T_{x_0}\mathbb{R}^n$. By construction, $\mathcal{V}_{\Sigma, x_0}^\ell$ depends only on at most the $(\ell - 1)$ derivatives of X_0, X_1, \dots, X_m at x_0 . Hence, if $\tilde{\Sigma} = (\{Y_0, Y_1, \dots, Y_m\}, U)$ is a control-affine system whose Taylor expansion up to order $k_n - 1$ at x_0 agrees with that of Σ , then $\mathcal{V}_{\tilde{\Sigma}, x_0}^{k_j} = \mathcal{V}_{\Sigma, x_0}^{k_j}$ for all $j \in \{1, \dots, n\}$. Hence, $\mathcal{V}_{\tilde{\Sigma}, x_0}^{k_1} + \mathcal{V}_{\tilde{\Sigma}, x_0}^{k_2} + \dots + \mathcal{V}_{\tilde{\Sigma}, x_0}^{k_n} = T_{x_0}\mathbb{R}^n$ and thus by Theorem 3.4, $\tilde{\Sigma}$ is also STLC from x_0 . \square

5. Examples. Let us illustrate the procedure in the proof of Theorem 4.1 with two known examples.

Example 5.1. The following single-input control-affine system Σ was considered by Stefani [19]. The state space is $M = \mathbb{R}^3$, $x_0 = 0 \in \mathbb{R}^3$, and the system vector fields are

$$X_0 = x_1 \frac{\partial}{\partial x_2} + x_1^3 x_2 \frac{\partial}{\partial x_3}, \quad X_1 = \frac{\partial}{\partial x_1}.$$

Applying the definition, it is straightforward to show that Σ is Δ -homogeneous with respect to the dilation $\Delta_s(x) = (sx_1, s^2x_2, s^6x_3)$. Hence, by Theorem 4.1, Σ is STLC from x_0 if and only if $\mathcal{V}_{x_0}^1 + \mathcal{V}_{x_0}^2 + \mathcal{V}_{x_0}^6 = T_{x_0}\mathbb{R}^3$. For $u \in U$ let $\xi_u = X_0 + uX_1$. One computes, using Theorem 3.6, that $\mathcal{V}_{x_0}^2 = \text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$. According to Theorem 4.1, to produce variations in the $\pm \frac{\partial}{\partial x_3}$ directions, we need to look at variations of order six. Following the proof of Theorem 4.1, let $\tau(s) = (a_1s, a_2s, a_3s)$ and let $\xi = (\xi_{u_1}, \xi_{u_2}, \xi_{u_3})$, with $a_1u_1 + a_2u_2 + a_3u_3 = 0$. Then $\text{ord}_{x_0}(\xi, \tau) \geq 2$ and one computes that

$$\left. \frac{d^2}{ds^2} \right|_{s=0} \Phi_{x_0}^\xi(\tau(s)) = (u_1a_1(a_1 + 2a_2 + a_3) + u_2a_2(a_2 + a_3)) \frac{\partial}{\partial x_2}$$

and so we set $u_2 = -\frac{a_1(a_1+2a_2+a_3)u_1}{a_2(a_2+a_3)}$ so that $\text{ord}(\xi, \tau) \geq 3$. Then one computes that the derivatives of $\Phi_{x_0}^\xi(\tau(s))$ of orders 3, 4, and 5 vanish at $s = 0$ and that the 6th derivative of $\Phi_{x_0}^\xi(\tau(s))$ at $s = 0$ equals

$$-\frac{30a_1^4(a_1 + a_2)(a_1 - a_3)(a_1 + a_2 + a_3)(a_1a_2 + 2a_1a_3 + a_2a_3)u_1^4}{(a_2 + a_3)^3} \frac{\partial}{\partial x_3}.$$

By inspection, the above expression can be made negative and positive for all choices of $u_1 \neq 0$ for appropriate values of $a_1, a_2, a_3 > 0$. Hence, $\text{span}\{\frac{\partial}{\partial x_3}\} \subset \mathcal{V}_{x_0}^6$. Moreover, because u_2 and u_3 are proportional to u_1 , we can make u_1 sufficiently small to force u_1, u_2, u_3 to lie in the interior of U . Hence, the system is STLC from x_0 by Theorem 4.1.

Example 5.2. The following single-input control-affine system Σ was considered in [14]. The state space is $M = \mathbb{R}^4$, $x_0 = 0 \in \mathbb{R}^4$, and the system vector fields are

$$X_0 = x_1 \frac{\partial}{\partial x_2} + \frac{1}{6}x_1^3 \frac{\partial}{\partial x_3} + x_2x_3 \frac{\partial}{\partial x_4}, \quad X_1 = \frac{\partial}{\partial x_1}.$$

Applying the definition, it is straightforward to verify that Σ is Δ -homogeneous with respect to the dilation $\Delta_s(x) = (sx_1, s^2x_2, s^4x_3, s^7x_4)$. Hence, by Theorem 4.1, Σ is STLC from x_0 if and only if $\mathcal{V}_{x_0}^1 + \mathcal{V}_{x_0}^2 + \mathcal{V}_{x_0}^4 + \mathcal{V}_{x_0}^7 = T_{x_0}\mathbb{R}^4$. For $u \in U$ let $\xi_u = X_0 + uX_1$. We proceed in the following steps:

- (i) Using Theorem 3.6, one computes that $\mathcal{V}_{x_0}^2 = \text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$.
- (ii) According to Theorem 4.1, to produce $\pm \frac{\partial}{\partial x_3}$ as variations, we must look at variations of order 4. Let $\tau(s) = (a_1s, a_2s, a_3s)$ and $\xi = (\xi_{u_1}, \xi_{u_2}, \xi_{u_3})$, where $a_1u_1 + a_2u_2 + a_3u_3 = 0$. Then $\text{ord}_{x_0}(\xi, \tau) \geq 2$ and

$$(5.1) \quad \left. \frac{d^2}{ds^2} \right|_{s=0} \Phi_{x_0}^\xi(\tau(s)) = (a_1^2u_1 + a_1(2a_2 + a_3)u_1 + a_2(a_2 + a_3)u_2) \frac{\partial}{\partial x_2}.$$

Setting $u_2 = -\frac{1}{a_2(a_2+a_3)}(a_1^2u_1 + a_1(2a_2 + a_3)u_1)$ results in $\text{ord}_{x_0}(\xi, \tau) \geq 4$ and

$$\left. \frac{d^4}{ds^4} \right|_{s=0} \Phi_{x_0}^\xi(\tau(s)) = -\frac{a_1^3(a_1 + a_2)(a_1 - a_3)(a_1 + a_2 + a_3)u_1^3}{(a_2 + a_3)^2} \frac{\partial}{\partial x_3}.$$

We can then vary the parameters $a_1, a_2, a_3 > 0$ to produce the variations $\pm \frac{\partial}{\partial x_3}$ for any $u_1 \neq 0$. Therefore $\text{span}\{\frac{\partial}{\partial x_3}\} \subset \mathcal{V}_{x_0}^4$.

- (iii) Now we investigate whether we can produce variations in the directions $\pm \frac{\partial}{\partial x_4}$. Let $\tau(s) = (a_1s, a_2s, a_3s)$ and let $\xi = (\xi_{u_1}, \xi_{u_2}, \xi_{u_3})$, where $a_1u_1 + a_2u_2 + a_3u_3 = 0$. If we set $\tilde{\xi} = (\xi_{-u_1}, \xi_{-u_2}, \xi_{-u_3})$, then $\text{ord}_{x_0}(\xi * \tilde{\xi}, \tau * \tau) \geq 3$ because the controls u_1, u_2, u_3 enter linearly in (5.1). In fact, one can compute that $\text{ord}_{x_0}(\xi * \tilde{\xi}, \tau * \tau) \geq 7$, and if we set $u_2 = \lambda u_1$, then

$$\left. \frac{d^7}{ds^7} \right|_{s=0} (\Phi_{x_0}^{\xi * \tilde{\xi}} \circ (\tau * \tau))(s) = f_a(\lambda)u_1^4 \frac{\partial}{\partial x_4},$$

where $f_a(\lambda)$ is a polynomial in λ of degree four with coefficients depending polynomially on $\mathbf{a} = (a_1, a_2, a_3)$. Choosing $\mathbf{a}^* = (1, 1/4, 10)$, we obtain that

$$f_{\mathbf{a}^*}(\lambda) = \frac{2007761}{16} + \frac{7105411}{64}\lambda + \frac{9990047}{256}\lambda^2 + \frac{6410283}{1024}\lambda^3 + \frac{6186859}{16384}\lambda^4.$$

One can verify that $f_{\mathbf{a}^*}(-5) < 0$ and that $f_{\mathbf{a}^*}(-4) > 0$. Hence, for any value of $u_1 \neq 0$, we can produce $\pm \frac{\partial}{\partial x_4}$ as a variation of order 7. Therefore $\text{span}\{\frac{\partial}{\partial x_4}\} \subset \mathcal{V}_{x_0}^7$. From the relationships $a_1u_1 + a_2u_2 + a_3u_3 = 0$ and $u_2 = \lambda u_1$, and the chosen \mathbf{a}^* , we obtain that $u_3 = -\frac{1}{40}(4 + \lambda)u_1$. Hence, by choosing u_1 sufficiently small, we can force $u_1, u_2, u_3 \in U$ since U contains the origin in its interior. Therefore, by Theorem 4.1, Σ is STLC from x_0 .

In the following example we consider a family of control-affine systems.

Example 5.3. Consider the control-affine system Σ on $M = \mathbb{R}^m \times \mathbb{R}^r$ of the form

$$(5.2) \quad \begin{aligned} \dot{x} &= u, \\ \dot{y} &= F(x), \end{aligned}$$

where $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^r$, $u \in U \subset \mathbb{R}^m$ and $F : \mathbb{R}^m \rightarrow \mathbb{R}^r$ is a homogeneous map of integer degree $k \geq 2$, that is, $F(\lambda x) = \lambda^k F(x)$ for all $x \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$. Let $X_0(x, y) = \sum_{j=1}^r F_j(x) \frac{\partial}{\partial y_j}$ denote the associated drift vector field, where we denote $F(x) = (F_1(x), \dots, F_r(x))$, and $X_1 = \frac{\partial}{\partial x_1}, \dots, X_m = \frac{\partial}{\partial x_m}$ the associated control vector fields of (5.2). For $u = (u_1, \dots, u_m) \in U$ let $\xi_u = X_0 + u_1X_1 + \dots + u_mX_m$. Let $z_0 = (0, 0) \in \mathbb{R}^m \times \mathbb{R}^r$.

Applying the definition, it is straightforward to verify that (5.2) is Δ -homogeneous with respect to the dilation $\Delta_s(x, y) = (sx, s^{k+1}y)$. For this system, it is clear that $\mathcal{V}_{z_0}^1 = \text{span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$, provided U contains the origin in its interior. (In fact all we need is that $\text{co}(U)$ contains the origin in its interior.) Hence, according to Theorem 4.1, (5.2) is STLC from the origin if and only if $\text{span}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_r}\} \subset \mathcal{V}_{z_0}^{k+1}$. A sufficient condition for $\text{span}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_r}\} \subset \mathcal{V}_{z_0}^{k+1}$ is that $\text{co}(\text{img}(F)) = \mathbb{R}^r$. To prove this, a straightforward but tedious calculation shows that if $\pm u \in U$, then

$$\left. \frac{d^{k+1}}{ds^{k+1}} \right|_{s=0} \Phi_s^{\xi^{-u}} \circ \Phi_s^{\xi^u}(z_0) = 2(k-1)! \sum_{j=1}^r F_j(u) \frac{\partial}{\partial y_j}.$$

Since we assume that U contains a neighbourhood of the origin and $\mathcal{V}_{z_0}^{k+1}$ is a convex cone, it follows that the convex hull of the set

$$\left\{ \sum_{j=1}^r F_j(x) \frac{\partial}{\partial y_j} : x \in \mathbb{R}^m \right\}$$

is contained in $\mathcal{V}_{z_0}^{k+1}$. Therefore, $\text{co}(\text{img}(F)) = \mathbb{R}^r$ implies that $\text{span}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_r}\} \subset \mathcal{V}_{z_0}^{k+1}$.

In the proof of Theorem 4.1, *linear* end-times were used. As we show in the next example, this can result in an over estimation for an integer k for which $\mathcal{V}_{x_0}^k = T_{x_0}\mathbb{R}^n$, i.e., the bound $\text{lcm}(k_1, \dots, k_n)$ in Corollary 4.2 is not sharp. This apparent inefficiency is an immaterial artifact of our decision to use smooth end-times and does not, for example, have any impact on our main theorems Theorem 4.1 and 4.3. The following example will make this point clear.

Example 5.4. We again consider the homogeneous system in Example 5.2, in which the integers associated with the dilation are $k_1 = 1, k_2 = 2, k_3 = 4, k_4 = 7$. In that example, we showed that $\text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\} \subset \mathcal{V}_{x_0}^4$. We now show, by using higher-order end-times, that $\text{span}\{\frac{\partial}{\partial x_4}\} \subset \mathcal{V}_{x_0}^8$, and thus by Lemma 3.2, $\mathcal{V}_{x_0}^8 = T_{x_0}\mathbb{R}^n$, while from Corollary 4.2 we can only conclude that $\mathcal{V}_{x_0}^{28} = T_{x_0}\mathbb{R}^n$. This apparent weakness has no impact on the efficiency of our approach to determine STLC from the derivatives of the system since from Theorem 4.3 any perturbation of order greater than 6 will not destroy STLC for this system, whereas the fact that $\mathcal{V}_{x_0}^8 = T_{x_0}\mathbb{R}^n$ allows one to conclude the weaker statement that any perturbation of order greater than 7 will not destroy STLC for this system.

For $u \in U$ let $\xi_u = X_0 + uX_1$. Producing a variation in the direction $\frac{\partial}{\partial x_4}$ is straightforward but we will treat both cases $\pm \frac{\partial}{\partial x_4}$ simultaneously. To this end, let $\tau_i(s) = a_i s + b_i \frac{s^2}{2}$ for $i = 1, 2, 3$, let $\tau(s) = (\tau_1(s), \tau_2(s), \tau_3(s))$, let $\xi = (\xi_{u_1}, \xi_{u_2}, \xi_{u_3})$, let $\tilde{\tau}(s) = (\tau_3(s), \tau_2(s), \tau_1(s))$, and let $\tilde{\xi} = (\xi_{u_3}, \xi_{u_2}, \xi_{u_1})$. If $a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$, then $\text{ord}_{x_0}(\xi, \tau) \geq 2$ and

$$\begin{aligned} \frac{d^2}{ds^2} \Big|_{s=0} \Phi_{x_0}^\xi(\tau(s)) &= \left(b_1 u_1 + b_2 u_2 - \frac{b_3(a_1 u_1 + a_2 u_2)}{a_3} \right) \frac{\partial}{\partial x_2} \\ &\quad + (a_1^2 u_1 + a_1(2a_2 + a_3)u_1 + a_2(a_2 + a_3)u_2) \frac{\partial}{\partial x_3}. \end{aligned}$$

If we set $b_3 = \frac{a_3}{a_1 u_1 + a_2 u_2} (b_1 u_1 + b_2 u_2)$, then we obtain that

$$(5.3) \quad \frac{d^2}{ds^2} \Big|_{s=0} \Phi_{x_0}^\xi(\tau(s)) = [(a_1^2 u_1 + a_1(2a_2 + a_3)u_1 + a_2(a_2 + a_3)u_2)] \frac{\partial}{\partial x_3}.$$

It is not hard to choose u_1, u_2, a_1, a_2 to make the tangent vector in (5.3) equal to zero so that we can continue to produce a higher-order variation. Instead, we augment to (ξ, τ) the reverse pair $(\tilde{\xi}, \tilde{\tau})$ so that we can keep the variables u_1, u_2, a_1, a_2 free and simultaneously cancel the tangent vector in (5.3). In fact, one computes that if we continue to use

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0 \quad \text{and} \quad b_3 = \frac{a_3}{a_1 u_1 + a_2 u_2} (b_1 u_1 + b_2 u_2),$$

then $\text{ord}_{x_0}(\xi * \tilde{\xi}, \tau * \tilde{\tau}) \geq 7$ and

$$\left. \frac{d^7}{ds^7} \right|_{s=0} \Phi_{x_0}^{\xi * \tilde{\xi}}((\tau * \tilde{\tau})(s)) = f_{\mathbf{a}}(u_1, u_2) \frac{\partial}{\partial x_4},$$

where $f_{\mathbf{a}}(u_1, u_2)$ is a homogeneous polynomial of degree 4 in the variables (u_1, u_2) whose coefficients are homogeneous polynomials in $\mathbf{a} = (a_1, a_2, a_3)$ of degree 7. Setting $\mathbf{a}^* = (1, 1/10, 5)$ and $u_2 = \lambda u_1$, where $\lambda \in \mathbb{R}$ is to be determined, one computes that

$$f_{\mathbf{a}^*}(u_1, \lambda u_1) = [c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + c_4\lambda^4] u_1^4,$$

where c_0, \dots, c_4 are positive rational numbers. Using a computer algebra system, one can verify that the polynomial $c(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + c_4\lambda^4$ has two real roots and they can be computed explicitly. Up to four digits they are given as $\lambda_1 = -15.7499\dots$ and $\lambda_2 = -13.4544\dots$. Hence, choosing $\mathbf{a}^* = (1, 1/10, 5)$ and $\lambda = \lambda_1$ yields that $\text{ord}_{x_0}(\xi * \tilde{\xi}, \tau * \tilde{\tau}) \geq 8$ and one computes that

$$\left. \frac{d^8}{ds^8} \right|_{s=0} \Phi_{x_0}^{\xi * \tilde{\xi}}((\tau * \tilde{\tau})(s)) = (-r_1 b_1 + r_2 b_2) u_1^4 \frac{\partial}{\partial x_4},$$

where $r_1, r_2 > 0$ are constants. By inspection, one can vary the parameters b_1 and b_2 to produce variations in the $\pm \frac{\partial}{\partial x_4}$ directions for any choice of $u_1 \neq 0$. Moreover, since u_2 and u_3 are proportional to u_1 , by choosing u_1 sufficiently small we can force $u_1, u_2, u_3 \in U$.

6. Conclusion. In this paper we considered the small-time local controllability problem for control-affine systems that are homogeneous with respect to a one-parameter family of dilations corresponding to time-scalings of the control. The main contribution was the identification of a relatively simple variational cone to characterize STLC for this important class of nonlinear control-affine systems. Although our main results do not give explicit computational conditions for STLC, they can potentially be used as a guide to develop sharp Lie bracket conditions for STLC for the systems in consideration.

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