Series solutions of HJB equations

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Abstract. We examine three methods for solving the Hamilton Jacobi Bellman PDE that arises in infinite horizon optimal control problems.

1 Introduction

The Hamilton Jacobi Bellman Partial Differential Equation (HJB PDE) characterizes the solution of an optimal control problem. Consider the problem of finding a control trajectory $u(t),\ 0 \le t \le \infty$ that minimizes the integral of a Lagrangian

$$\int_0^\infty l(x,u)\, dt$$

subject to the dynamic constraints

$$\dot{x} = f(x,u), \qquad x(0) = x^0$$

where $x \in \mathbb{R}^{n\times 1},\ u \in \mathbb{R}^{m\times 1}$.

If $f,\ l$ are smooth and the optimal cost is a smooth function $\pi(x^0)$ of the initial condition then the optimal control is given by an optimal feedback $u(t) = \kappa(x(t))$ and the HJB PDE is satisfied,

$$0 = \min_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\},$$

$$\kappa(x) \in \operatorname*{argmin}_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\}.$$

If we further assume that the control Hamiltonian

$$H(\lambda,x,u) = \lambda f(x,u) + l(x,u)$$

is strictly convex in u for all $\lambda \in \mathbb{R}^{1\times n}$ and $x \in \mathbb{R}^{n\times 1}$ then the HJB PDE can be rewritten as

$$0 = \frac{\partial \pi}{\partial x}(x) f(x,\kappa(x)) + l(x,\kappa(u)), \tag{1}$$

$$0 = \frac{\partial \pi}{\partial x}(x) \frac{\partial f}{\partial u}(x,\kappa(x)) + \frac{\partial l}{\partial u}(x,\kappa(u)). \tag{2}$$

The simplest example of this is the so called Linear Quadratic Regulator (LQR) where the dynamics is linear and the Lagrangian is quadratic

$$f(x,u) = Fx + Gu, \qquad l(x,u) = \frac{1}{2}\left(x^\top Qx + 2x^\top Su + u^\top Ru\right)$$

where

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$$

is nonnegative definite and R is positive definite. If $F,\ G$ is stabilizable and $Q^{\frac{1}{2}},\ F$ is detectable then the HJB PDE has a unique solution

$$\pi(x) = \frac{1}{2}x^\top Px, \qquad \kappa(x) = Kx$$

where P is the unique nonnegative definite solution of the algebraic Riccati equation

$$F^\top P + PF + Q - (PG+S)R^{-1}(PG+S)^\top = 0 \tag{3}$$

and

$$K = -R^{-1}(PG+S)^\top. \tag{4}$$

Moreover all the eigenvalues of $F+GK$ are in the open left half plane so the closed loop dynamics

$$\dot{x} = (F+GK)x \tag{5}$$

is exponentially stable.

We return to the nonlinear problem. Perhaps the principle reason for trying to solve an optimal control problem is to find a feedback $u = \kappa(x)$ that makes the closed loop system

$$\dot{x} = f(x,\kappa(x)) \tag{6}$$

asymptotically stable. If the HJB PDE can be solved for $\pi(x),\ \kappa(x)$ then the closed loop system can be shown to be asymptotically stable in some region around the origin by a Lyapunov argument,

$$\frac{d}{dt}\pi(x(t)) = \frac{\partial\pi}{\partial x}(x(t))f(x(t),\kappa(x(t))) = -l(x(t),\kappa(u(t))) \le 0.$$

Given an approximate solution $\pi(x)$ to the HJB PDE we seek the largest punctured sublevel set of $\pi(x)$ where $\pi(x) > 0$ and $\frac{\partial\pi}{\partial x}(x)f(x,\kappa(x)) < 0$. Then we know that this punctured sublevel set is in the basin of attraction of the origin for the closed loop dynamics (6).

Suppose the Lagrangian and the dynamics have Taylor series expansions

$$f(x,u) = Fx + Gu + f^{[2]}(x,u) + f^{[3]}(x,u) + \dots, \tag{7}$$

$$l(x,u) = \frac{1}{2}\left(x^\top Qx + u^\top Ru\right) + l^{[3]}(x,u) + l^{[4]}(x,u) + \dots, \tag{8}$$

where $^{[d]}$ denotes polynomial vector fields homogeneous of degree d in $x,\ u$.

Various methods have been proposed in the literature ([2] and references) to find similar series expansions of the optimal cost and/or the stabilizing optimal feedback,

$$\pi(x) = \frac{1}{2}x^\top Px + \pi^{[3]}(x) + \pi^{[4]}(x) + \dots, \tag{9}$$

$$\kappa(x) = Kx + \kappa^{[2]}(x) + \kappa^{[3]}(x) + \dots. \tag{10}$$

We shall examine three of them, Al'brecht's method [1], the state dependent Riccati equation method [3, 4] and Garrard's method [5–7]. We shall describe these methods and see how well they do on a simple example.

This paper is dedicated to our esteemed colleague and good friend Uwe Helmke on the occasion of his sixtieth birthday.

2 Al'brecht's Method

Al'brecht's method has been discussed and used in [9, 11, 13] and many other papers. Al'brecht plugged the series expansions (7–10) into the HJB equations (1, 2) and collected terms degree by degree. The lowest terms of the first HJB equation (1) are of degree 2 and the lowest terms of the second HJB equation (2) are of degree 1. They reduce to the algebraic Riccati equation (3) and the formula for the linear gain (4). Therefore Al'brecht assumed that $F,\ G,\ Q,\ R,\ S$ satisfied the assumptions of the Linear Quadratic Regulator discussed above so that these equations have a unique solution.

Having found $P,\ K$ we turn to the degree 3 terms of (1) and the degree 2 terms of (2),

$$0 = \frac{\partial \pi^{[3]}}{\partial x}(x)(F+GK)x + x^\top P f^{[2]}(x,Kx) + l^{[3]}(x,Kx), \tag{11}$$

$$0 = \frac{\partial \pi^{[3]}}{\partial x}(x)G + x^\top P\frac{\partial f^{[2]}}{\partial u}(x,Kx) + \frac{\partial l^{[3]}}{\partial u}(x,Kx) + \left(\kappa^{[2]}(x)\right)^\top R. \tag{12}$$

The unknowns in these equations are $\pi^{[3]}(x)$ and $\kappa^{[2]}(x)$ and the equations are triangular, the second unknown does not appear in the first equation. To decide the solvability of the first, we study the linear operator

$$\pi^{[3]}(x) \mapsto \frac{\partial \pi^{[3]}}{\partial x}(x)(F+GK)x$$

from cubic polynomials to cubic polynomials. Its eigenvalues are of the form $\lambda_i + \lambda_j + \lambda_k$ where $\lambda_i,\ \lambda_j,\ \lambda_k$ are eigenvalues of $F+GK$. A cubic resonance occurs when such a sum equals zero. But all the eigenvalues of $F+GK$ are in the open left half plane so there are no cubic resonances.

Hence there is a unique solution to the first equation for $\pi^{[3]}(x)$ and then the second equation yields

$$\kappa^{[2]}(x) = -R^{-1}\left(\frac{\partial \pi^{[3]}}{\partial x}(x)G + x^\top P\frac{\partial f^{[2]}}{\partial u}(x,Kx) + \frac{\partial l^{[3]}}{\partial u}(x,Kx)\right)^\top$$

Then we find $\pi^{[4]}(x)$ from the degree 4 terms in (1)

$$0 = \frac{\partial \pi^{[4]}}{\partial x}(x)(F+GK)x + \frac{\partial \pi^{[3]}}{\partial x}(x)\left(f(x,Kx+\kappa^{[2]}(x))\right)^{[2]} + x^{\mathsf{T}} P\left(f(x,Kx+\kappa^{[2]}(x))\right)^{[3]} + l^{[4]}(x,Kx) + \left(l^{[3]}(x,Kx+\kappa^{[2]}(x))\right)^{[4]}$$

where $(\cdot)^{[d]}$ denotes the degree d part of the expression in the parenthesis. This equation is always solvable because the map

$$\pi^{[4]}(x) \mapsto \frac{\partial \pi^{[4]}}{\partial x}(x)(F+GK)x$$

from quartic polynomials to quartic polynomials has eigenvalues of the form $\lambda_i + \lambda_j + \lambda_k + \lambda_l$ where the λ's are eigenvalues of $F+GK$. Then the degree 3 part of (2) yields

$$\kappa^{[3]}(x) = -R^{-1}\left(\frac{\partial \pi^{[4]}}{\partial x}(x)G + \frac{\partial \pi^{[3]}}{\partial x}(x)\frac{\partial f^{[2]}}{\partial u}(x,Kx) + x^{\mathsf{T}} P\left(\frac{\partial f^{[3]}}{\partial u}(x,Kx+\kappa^{[2]}(x))\right)^{[2]} + \frac{\partial l^{[3]}}{\partial u}(x,Kx)\right)^{\mathsf{T}}.$$

The higher degree terms are found in a similar fashion. The MATLAB based Nonlinear Systems Toolbox [8] that was written by one of authors contains a routine "hjb.m" that implements Al'brecht method. It runs very fast when n, m, d are small to medium. For example when $n=6$, $m=3$, $d=3$ the routine takes 0.076734 seconds on a MacBook Pro with an 2.66 GHz Intel Core Duo processor. When d is increased to 5 it takes 3.422941 seconds.

Al'brecht's method generates a candidate Lyapunov function $\pi(x)$ for closed loop dynamics

$$\dot{x} = f(x,\kappa(x))$$

because

$$\begin{aligned}\frac{d}{dt}\pi(x(t)) &= \frac{\partial \pi}{\partial x}(x(t))f(x(t),\kappa(x(t))) \\ &= -l(x(t),\kappa(x(t))) + O(x(t))^{d+2}\end{aligned}$$

One seeks the largest sub level set $\{x : \pi(x) \le c\}$ where for $x \neq 0$

$$\begin{aligned}\pi(x) &> 0, \\ \frac{\partial \pi}{\partial x}(x)f(x,\kappa(x)) &< 0.\end{aligned} \tag{13}$$

In the second inequality the true $f(x,u)$ should be used, not its Taylor expansion. The second inequality can be relaxed using the LaSalle invariance principle.

A modification of Al'brecht's method can also be used to generate a candidate Lyapunov function for an uncontrolled dynamics

$$\dot{x} = f(x) = Fx + f^{[2]}(x) + f^{[3]}(x) + \dots.$$

The first step is to solve for P a linear Lyapunov equation of the form

$$FP + PF + Q = 0,$$

where Q is chosen to be positive definite. The candidate Lyapunov function is

$$\pi(x) = \frac{1}{2}x^\top Px + \pi^{[3]}(x) + \pi^{[4]}(x) + \dots,$$

where π is the solution of the nonlinear Lyapunov equation

$$0 = \frac{\partial \pi}{\partial x}(x) f(x) + \frac{1}{2}x^\top Qx.$$

This equation is a degenerate HJB equation with no control and so it can also be solved term by term. The method is due to Zubov [14] and is implemented by "zbv.m" in the Nonlinear Systems Toolbox.

3 State Dependent Riccati Equation Method

The state dependent Riccati equation (SDRE) method can be used on problems of the form

$$\begin{aligned} f(x,u) &= F(x)x + G(x)u, \\ l(x,u) &= \frac{1}{2}\left(x^\top Q(x)x + 2x^\top S(x)u + u^\top R(x)u\right). \end{aligned}$$

Many nonlinear optimal control problems can be written in this form. To do so the only additional restrictions on (7, 8) are that the dynamics $f(x,u)$ be linear in u and the Lagrangian be quadratic in u. Usually there are many different ways to choose $F(x)$, $G(x)$, $Q(x)$, $R(x)$, $S(x)$ and little seems to be known about which choices are better than others.

One assumes that the optimal cost and optimal feedback have similar nonunique representations

$$\pi(x) = \frac{1}{2}x^\top P(x)x, \qquad \kappa(x) = K(x)x.$$

Then the HJB equations become

$$\begin{aligned} 0 = x^\top \Big(& F^\top(x)P(x) + P(x)F(x) + Q(x) \\ & -(P(x)G(x) + S(x))R^{-1}(x)(P(x)G(x) + S(x))^\top \Big) x \\ & + \sum_{ij} \frac{\partial P_{ij}}{\partial x}(x)\left(F(x)x + G(x)K(x)x\right)x_i x_j, \\ 0 = x^\top & (P(x)G(x) + S(x)) + x^\top K^\top(x)R(x) \\ & + \sum_{ij} \frac{\partial P_{ij}}{\partial x}(x)G(x)x_i x_j. \end{aligned}$$

In the SDRE method one ignores the last sum in each of these equations to obtain

$$\begin{aligned}0 &= x^\top \big(F^\top(x)P(x) + P(x)F(x) + Q(x)\\ &\qquad -(P(x)G(x)+S(x))R^{-1}(x)(P(x)G(x)+S(x))^\top\big)x,\\ 0 &= x^\top(P(x)G(x)+S(x)) + x^\top K^\top(x)R(x),\end{aligned}$$

which reduce to the state dependent Riccati equation and a formula for the state dependent gain

$$\begin{aligned}0 &= F^\top(x)P(x) + P(x)F(x) + Q(x)\\ &\qquad -(P(x)G(x)+S(x))R^{-1}(x)(P(x)G(x)+S(x))^\top, \qquad (14)\\ K(x) &= -R^{-1}(x)(P(x)G(x)+S(x))^\top. \qquad (15)\end{aligned}$$

To our knowledge the mathematical justification for omitting the last sums has never been clearly explained. But the result is to replace a nonlinear partial differential equation (HJB) with a nonlinear functional equation (SDRE). Whether this is a true simplification is questionable. There have been several recommendations about how to solve SDRE [3]. A symbolic software package such as Maple or Mathematica may be able to solve simple systems with special structure. Another possibility is to solve it online at a relatively high bit rate. Or perhaps it can be solved offline at a large number of states and then gain scheduling is used in between. In [12] an equation similar to the SDRE is solved by series expansion in a small parameter.

We shall show that it can also be solved by series expansion in the state vector. Assume there are the following series expansions.

$$\begin{aligned}F(x) &= F^{[0]} + F^{[1]}(x) + F^{[2]}(x) + \dots,\\ G(x) &= G^{[0]} + G^{[1]}(x) + G^{[2]}(x) + \dots,\\ Q(x) &= Q^{[0]} + Q^{[1]}(x) + Q^{[2]}(x) + \dots,\\ R(x) &= R^{[0]} + R^{[1]}(x) + R^{[2]}(x) + \dots, \qquad (16)\\ S(x) &= S^{[0]} + S^{[1]}(x) + S^{[2]}(x) + \dots,\\ P(x) &= P^{[0]} + P^{[1]}(x) + P^{[2]}(x) + \dots,\\ K(x) &= K^{[0]} + K^{[1]}(x) + K^{[2]}(x) + \dots,\end{aligned}$$

where the superscript $^{[d]}$ denotes a matrix valued polynomial that is homogeneous of degree d in x.

The first step is to expand $R^{-1}(x)$. It is not hard to verify that

$$R^{-1}(x) = T^{[0]} + T^{[1]}(x) + T^{[2]}(x) + \dots,$$

where

$$\begin{aligned}T^{[0]} &= (R^{[0]})^{-1},\\ T^{[1]}(x) &= -(R^{[0]})^{-1}R^{[1]}(x)(R^{[0]})^{-1},\\ T^{[2]}(x) &= -(R^{[0]})^{-1}R^{[2]}(x)(R^{[0]})^{-1} + (R^{[0]})^{-1}R^{[1]}(x)(R^{[0]})^{-1}R^{[1]}(x)(R^{[0]})^{-1}.\end{aligned}$$

If we plug these expansions into (14, 15) and collect the degree 0 terms we get the familiar algebraic Riccati and gain equations for the linear quadratic part of the problem

$$0 = (F^{[0]})^\top P^{[0]} + P^{[0]}F^{[0]} + Q^{[0]} - (P^{[0]}G^{[0]} + S^{[0]})T^{[0]}(P^{[0]}G^{[0]} + S^{[0]})^\top_; \quad (17)$$

$$K^{[0]} = -T^{[0]}(P^{[0]}G^{[0]} + S^{[0]})^\top. \quad (18)$$

Having solved these equations for $P^{[0]}$, $K^{[0]}$ we collect the terms of degree 1 in (14, 15),

$$\begin{aligned} 0 = {} & (F^{[0]} + G^{[0]}K^{[0]})^\top P^{[1]}(x) + P^{[1]}(x)(F^{[0]} + G^{[0]}K^{[0]}) \\ & + (F^{[1]}(x))^\top P^{[0]} + P^{[0]}F^{[1]}(x) + Q^{[1]}(x) \\ & - (P^{[0]}G^{[1]}(x) + S^{[1]})T^{[0]}(P^{[0]}G^{[0]}(x) + S^{[0]})^\top \cdot \qquad (19) \\ & \quad \cdot (P^{[0]}G^{[0]}(x) + S^{[0]})T^{[0]}(P^{[0]}G^{[1]}(x) + S^{[1]})^\top \\ & - (P^{[0]}G^{[0]}(x) + S^{[0]})T^{[1]}(x)(P^{[0]}G^{[0]}(x) + S^{[0]})^\top, \end{aligned}$$

$$K^{[1]}(x) = -T^{[0]}(P^{[1]}G^{[0]}(x)P^{[0]}G^{[1]} + S^{[10]})^\top - T^{[1]}(x)(P^{[0]}G^{[0]}(x) + S^{[0]})^\top_. \quad (20)$$

Notice that (19) is a linear Lyapunov equation in the unknown $P^{[1]}(x)$. If all the eigenvalues of $F^{[0]} + G^{[0]}K^{[0]}$ are in the open left half plane then this equation is always solvable because the eigenvalues of

$$P^{[1]}(x) \mapsto (F^{[0]} + G^{[0]}K^{[0]})^\top P^{[1]}(x) + P^{[1]}(x)(F^{[0]} + G^{[0]}K^{[0]})$$

are sums of pairs of eigenvalues of $F^{[0]} + G^{[0]}K^{[0]}$ and so none of them can be zero if the linear quadratic part of the problem satisfies the standard LQR assumptions. The higher degree terms are found in a similar fashion.

The SDRE method also yields a candidate Lyapunov function $x^\top P(x)x$ for the closed loop dynamics

$$\dot{x} = (F(x) + G(x)K(x))x.$$

The Lyapunov derivative is

$$\begin{aligned} \frac{d}{dt}x^\top(t)P(x(t))x(t) = {} & x^\top(t)\,(F(x(t)) + G(x(t))K(x(t)))^\top P(x(t)) \\ & + P(x(t))\,(F(x(t)) + G(x(t))K(x(t)))\,x(t) \\ & + \sum_{ij} \frac{\partial P_{ij}}{\partial x}(x(t))\,(F(x(t)) + G(x(t))K(x(t)))\,x(t)x_i(t)x_j(t), \end{aligned}$$

which reduces to

$$\begin{aligned} \frac{d}{dt}x^\top(t)P(x(t))x(t) = {} & -x^\top(t)\big(Q(x(t)) + (P(x(t))G(x(t)) + S(x(t)))\,\cdot \\ & \cdot R^{-1}(x(t))(P(x(t))G(x(t)) + S(x(t)))^\top\big)x(t) \\ & + \sum_{ij} \frac{\partial P_{ij}}{\partial x}(x(t))\,(F(x(t)) + G(x(t))K(x(t)))\,x(t)x_i(t)x_j(t). \end{aligned}$$

So the quadratic part of the Lyapunov derivative is nonpositive but the higher terms may be positive.

4 Garrard's Method

Garrard's method is a simplification of Al'brecht's method that was developed when computing resources were more limited. Garrard considered a reduced set of problems where

$$l(x,u) = \frac{1}{2}\left(x^\top Q x + u^\top R u\right), \tag{21}$$

$$f(x,u) = Fx + Gu + f^{[d]}(x), \tag{22}$$

where d is either 2 or 3. His method does not yield an approximation to the optimal cost but it does yield an approximation to the optimal feedback,

$$\kappa(x) = Kx + \kappa^{[d]}(x).$$

As with all the series methods that we consider, Garrard assumed that F, G, Q, R satisfied the assumptions of the Linear Quadratic Regulator discussed above. The first step of the method is to find P, K as before.

Suppose $d = 2$, the next step is to solve (11). He rewrote this equation assuming (21, 22) as

$$0 = \left(\frac{\partial \pi^{[3]}}{\partial x}(x)(F+GK) + (f^{[2]}(x))^\top P\right)x \tag{23}$$

and ignored the fact that $\frac{\partial \pi^{[3]}}{\partial x}(x)$ is the gradient of a function. He treated it as an arbitrary row vector valued polynomial homogeneous of degree 2. Then (23) has multiple solutions. Since $F+GK$ is invertible one simple solution of (23) is

$$\frac{\partial \pi^{[3]}}{\partial x}(x) = -(f^{[2]}(x))^\top P(F+GK)^{-1}, \tag{24}$$

but this is usually not the gradient of a function because its mixed partials do not commute

$$\frac{\partial^2 \pi^{[3]}}{\partial x_i \partial x_j}(x) \neq \frac{\partial^2 \pi^{[3]}}{\partial x_j \partial x_i}(x).$$

Garrard set

$$\kappa^{[2]}(x) = -R^{-1}\left(\frac{\partial \pi^{[3]}}{\partial x}(x)G\right)^\top. \tag{25}$$

When $d = 3$ then $\pi^{[3]}(x) = 0,\ \kappa^{[2]}(x) = 0$ and the relevant equation is

$$0 = \left(\frac{\partial \pi^{[4]}}{\partial x}(x)(F+GK) + (f^{[3]}(x))^\top P\right)x.$$

Again if we ignore the fact that $\frac{\partial \pi^{[4]}}{\partial x}(x)$ is a gradient then one solution of (24) and (25) is

$$\frac{\partial \pi^{[4]}}{\partial x}(x) = -(f^{[3]}(x))^\top P(F+GK)^{-1},$$
$$\kappa^{[3]}(x) = -R^{-1}\left(\frac{\partial \pi^{[4]}}{\partial x}(x)G\right)^\top.$$

As we mentioned above Garrard only used his method to solve problems with one degree of nonlinearity in the dynamics but the method can be easily generalized to problems with multiple degrees of nonlinearity provided they are in the SDRE form (16).

We can solve for $\frac{\partial \pi^{[3]}}{\partial x}(x)$ by ignoring the fact that it is a gradient, cf. (24), and then use it to define $\kappa^{[2]}(x)$, cf. (25). We put $\kappa^{[2]}(x)$ in the form

$$\kappa^{[2]}(x) = K^{[1]}(x)x,$$

where $K^{[1]}(x)$ is an $m \times n$ matrix valued polynomial homogeneous of degree 1. Again this can always be done, usually in many ways.

At the next level the relevant equation is

$$0 = \left(\frac{\partial \pi^{[4]}}{\partial x}(x)(F^{[0]}+G^{[0]}K^{[0]}) + \frac{\partial \pi^{[3]}}{\partial x}(x)\left(F^{[1]}(x)+G^{[0]}K^{[1]}(x)\right) + (F^{[2]}(x)x)^\top P^{[0]}\right)x.$$

If we ignore the fact that $\frac{\partial \pi^{[4]}}{\partial x}(x)$ is a gradient this has a solution

$$\frac{\partial \pi^{[4]}}{\partial x}(x) = -\left(\frac{\partial \pi^{[3]}}{\partial x}(x)\left(F^{[1]}(x)+G^{[0]}K^{[1]}(x)+G^{[1]}(x)K^{[0]}\right) + x^\top (F^{[2]}(x))^\top P^{[0]}\right)(F+GK)^{-1}.$$

This can be continued to higher degrees but there is one significant disadvantage of this method. The function $\pi(x)$ is never computed so we don't have a potential Lyapunov function to check the basin of attraction of the closed loop system. One way around this is given the closed loop dynamics, use Zubov's method to compute a candidate Lyapunov function to determine the basin of attraction. But Zubov's method is a simplification of Al'brecht's method so why not just use Al'brecht's method?

5 Example

We apply the three methods described above to a simple problem where we know the exact solution. Consider the LQR problem of minimizing

$$\frac{1}{2}\int_0^\infty |z|^2 + u^2 \, dt$$

subject to

$$\dot z_1 = z_2,$$
$$\dot z_2 = u.$$

The optimal cost and optimal feedback are

$$\pi(z) = \frac{1}{2} z^{\top} \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} z,$$
$$u = -\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix} z.$$

If we make the nonlinear change of coordinates

$$z_1 = \sin x_1,$$
$$z_2 = x_2 - \frac{x_1^3}{3},$$

then the problem becomes nonlinear, minimize

$$\frac{1}{2}\int_0^\infty \sin^2 x_1 + \left(x_2 - \frac{x_1^3}{3}\right)^2 + u^2 \ \mathrm{dt}$$

subject to

$$\begin{aligned} \dot x_1 &= \left(x_2 - \frac{x_1^3}{3}\right)\sec x_1, \\ \dot x_2 &= \left(x_1^2 x_2 - \frac{x_1^5}{3}\right)\sec x_1 + u. \end{aligned} \tag{26}$$

But we know the true solution,

$$\begin{aligned} \pi(x) &= \frac{\sqrt{3}}{2}\sin^2 x_1 + \left(x_2 - \frac{x_1^3}{3}\right)\sin x_1 + \frac{\sqrt{3}}{2}\left(x_2 - \frac{x_1^3}{3}\right)^2, \\ \kappa(x) &= -\sin x_1 - \sqrt{3}\left(x_2 - \frac{x_1^3}{3}\right). \end{aligned} \tag{27}$$

Notice that the change of coordinates is a nonsingular mapping from $-\frac{\pi}{2} < x_1 < \frac{\pi}{2}$, $-\infty < x_2 < \infty$ to $-1 < z_1 < 1$, $-\infty < z_2 < \infty$. The nonlinear system (26) is only defined on the strip $-\frac{\pi}{2} < x_1 < \frac{\pi}{2}$, $-\infty < x_2 < \infty$ even though (27) defines $\pi(x)$ and $\kappa(x)$ on $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$.

We applied the power series methods described above to this nonlinear problem. For Al'brecht's method and the SDRE method we computed $\pi(x)$ to degree 4 and $\kappa(x)$ to degree 3. For the SDRE method

$$\pi(x) = x^{\top} P(x) x.$$

For Garrard's method we first computed $\kappa(x)$ to degree 3 and then found $\pi(x)$ of degree four by Zubov's method. Here are the results.

Method	Time (sec)	Norm π Error	Norm κ Error
Al'brecht	0.0090	$1.1771e-15$	$1.3476e-15$
SDRE	0.0136	0.4707	0.8951
Garrard	0.0154	$7.4470e-16$	1.8735

The times are essentially the same for the three methods. The π errors are the l_2 norms of the differences between the vectors of coefficients of the computed π's and the Taylor polynomial of degree 4 of the true π. The κ errors are the l_2 norms of the differences between the vectors of coefficients of the computed κ's and the Taylor polynomial of degree 3 of the true κ. The Al'brecht method computes the polynomials π and κ essentially to machine precision. The SDRE method makes substantial errors in both. Garrard's method also makes a substantial error in the computation of κ but π, computed by Zubov's method, corrects this error to machine precision. It is an open question whether this always happens.

Perhaps more important are the sizes of the basin of attraction of the closed loop dynamics of the three methods. So we computed these basins as follows. We plugged each third degree polynomial $\kappa(x)$ into the nonlinear dynamics (26) and computed the largest sublevel set of the corresponding fourth degree polynomial $\pi(x)$ where $\pi(x) \geq 0$ and the Lyapunov derivative of $\pi(x)$ is nonpositive. The results are shown in the figures on the following pages. The Al'brecht and Garrard basins of attraction appear identical perhaps because the corresponding π's are nearly equal while the SDRE basin of attraction is considerably smaller. It is perhaps a surprise that all of these basins are relatively small. After all, the LQR feedback globally stabilizes the linear system. So we computed the basin of attraction for Al'brecht's method where the optimal cost is computed to degree 6 and the optimal feedback is computed to degree 5. The computation took 0.210 seconds and the basin of attraction is shown in Figure 4. Notice the different scale from the other figures.

6 Conclusion

We have discussed three power series methods for approximately solving the HJB equation that arises in the infinite horizon optimal control problem. The computational burdens are roughly equivalent but only Al'becht's method can be mathematically justified. Therefore we recommend it.

We have seen in an example even when the Taylor polynomials of the optimal cost and optimal feedback are computed to machine precision, the closed loop dynamics may fail to have a large basin of attraction. This is because of the truncation of the higher order terms. One can increase the degree of the Taylor approximations but this does not always lead to a larger basin of attraction. Therefore we are developing patchy methods to remedy this [10].

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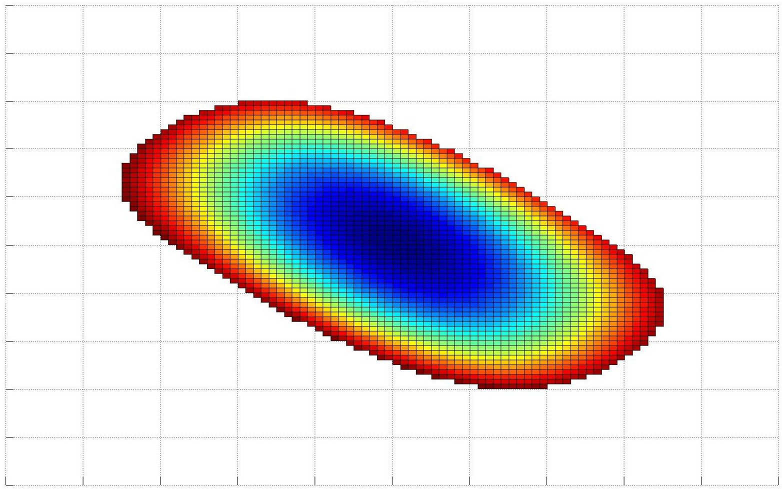

Figure 1: Al'brecht Basin of Attraction with $d = 3$, the region shown is $-1 \leq x_2 \leq 1$ on the vertical axis and $-1 \leq x_1 \leq 1$ on the horizontal axis.

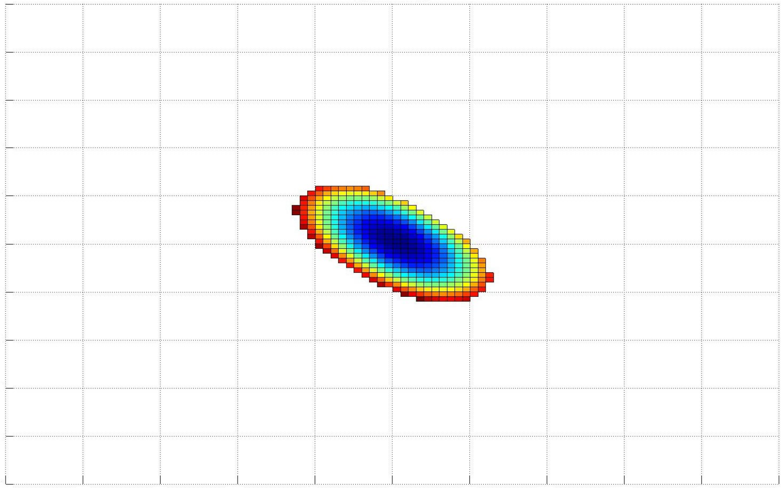

Figure 2: SDRE Basin of Attraction with $d = 3$, the region shown is $-1 \leq x_2 \leq 1$ on the vertical axis and $-1 \leq x_1 \leq 1$ on the horizontal axis.

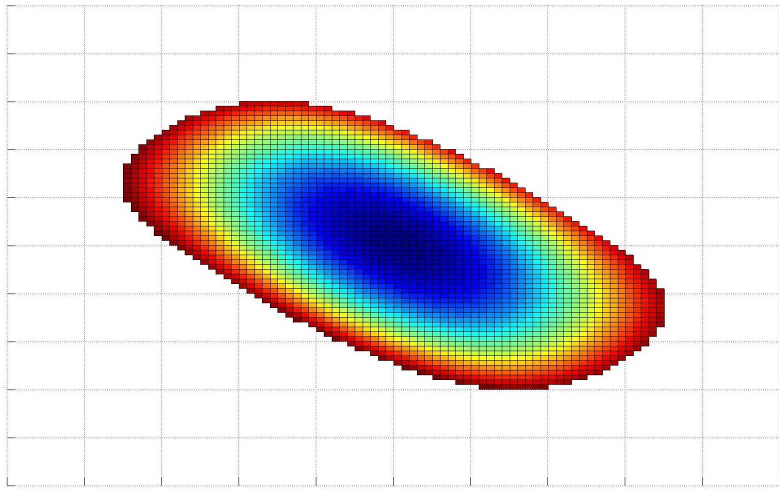

Figure 3: Garrard Basin of Attraction with $d = 3$, the region shown is $-1 \leq x_2 \leq 1$ on the vertical axis and $-1 \leq x_1 \leq 1$ on the horizontal axis.

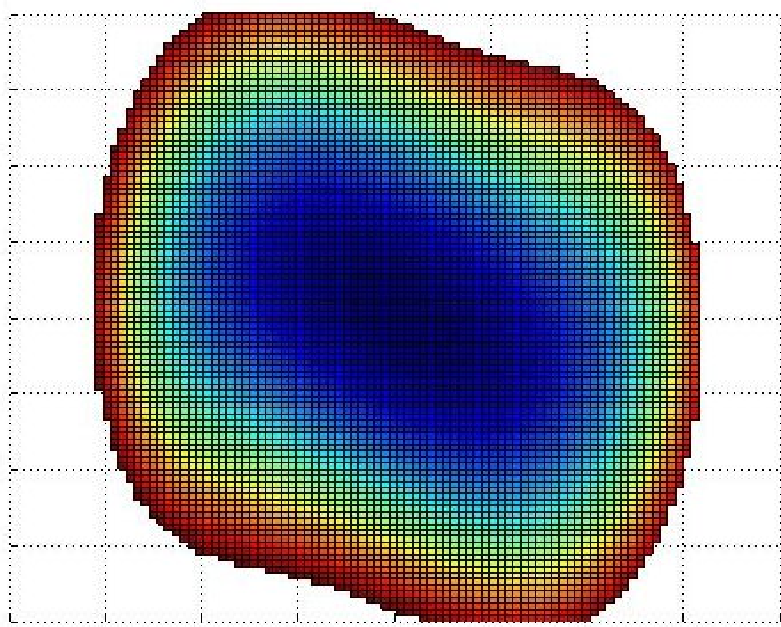

Figure 4: Al'brecht Basin of Attraction with $d = 5$, the region shown is $-2 \leq x_2 \leq 2$ on the vertical axis and $-2 \leq x_1 \leq 2$ on the horizontal axis.

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