Problem Set 2 Solution — Limits

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Math 221 05

Problem 1. Use algebra to find

$$\lim_{x \to \frac{\pi}{2}} (\cos x \tan x)$$

You will probably have to use your intuition about a certain limit at some point in solving this problem, but you can rewrite the original limit algebraically before you need that intuition.

Solution. Use the definition of $\tan x$, namely $\tan x = \frac{\sin x}{\cos x}$ to simplify the limit:

$$\begin{aligned} \lim_{x \to \frac{\pi}{2}} (\cos x \tan x) &= \lim_{x \to \frac{\pi}{2}} (\cos x \frac{\sin x}{\cos x}) \\ &= \lim_{x \to \frac{\pi}{2}} \sin x \\ &= \sin(\frac{\pi}{2}) \\ &= 1 \end{aligned}$$

Problem 2. Find

$$\lim_{x \to 1^-} \frac{x^2 - 1}{|x - 1|}$$

and

$$\lim_{x \to 1^+} \frac{x^2 - 1}{|x - 1|}$$

Solution. To find either limit, we need to get rid of the $|x-1|$ in the denominators of the fractions, because it goes to 0 when $x = 1$. Notice that the $x^2 - 1$ in the numerators factors to $(x-1)(x+1)$, which looks like a way to cancel out the $x-1$ in the denominators. Sadly, however, you cannot cancel $|x - 1|$ with $(x - 1)$ — the absolute value operator hides sign information that affects whether canceling produces 1 or -1. To solve this problem, realize that when $x < 1$, as when x approaches 1 from the left, $x - 1$ is negative, and so $|x - 1| = -(x - 1)$. Similarly, when $x > 1$, as when approaching 1 from the right, $x - 1 > 1$ and $|x - 1| = x - 1$. In short, we can replace $|x-1|$ in each limit with an equivalent expression for that limit, and cancel:

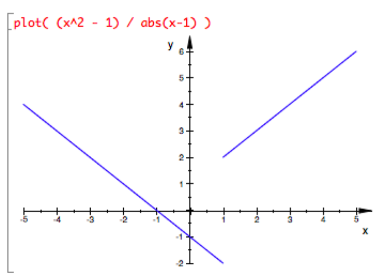
$$\begin{aligned}
\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} &= \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)} \\
&= \lim_{x\to 1^-} \frac{(x+1)(x-1)}{-(x-1)} \\
&= \lim_{x\to 1^-} -(x+1) \\
&= -2
\end{aligned}$$

The other side is similar:

$$\begin{aligned}
\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} &= \lim_{x\to 1^+} \frac{x^2-1}{x-1} \\
&= \lim_{x\to 1^+} \frac{(x+1)(x-1)}{x-1} \\
&= \lim_{x\to 1^+} (x+1) \\
&= 2
\end{aligned}$$

Use muPad to plot $y = \frac{x^2-1}{|x-1|}$ near $x = 1$ and verify that what you see is consistent with the limit(s) you calculated.

Solution. Use the muPad command `plot( (x^2 - 1) / abs(x - 1) )`. The result looks like the following, showing y reaching different limits for x values to the right of 1 and x values to the left of 1:



Does $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$ exist? If so, what is it? If not, why not?

Solution. Because the one-sided limits are unequal, the two-sided limit $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$ does not exist.

Problem 3. Find

$$\lim_{x\to 3} \frac{1}{x^2-2x-3}$$

Solution. We want to put the limit into a form to which one of the limit laws we've studied applies. Factoring the denominator offers one way to do this, namely

...

$$\begin{aligned}
\lim_{x\to 3} \frac{1}{x^2 - 2x - 3} &= \lim_{x\to 3} \frac{1}{(x+1)(x-3)} \\
&= \lim_{x\to 3} \frac{1}{x+1} \times \lim_{x\to 3} \frac{1}{x-3} \\
&= \frac{1}{4} \lim_{x\to 3} \frac{1}{x-3}
\end{aligned}$$

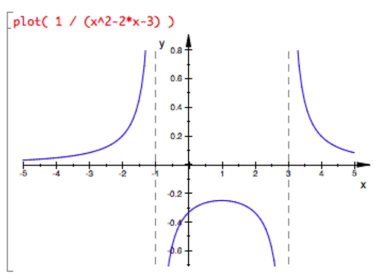
The remaining limit is one to which a limit law for infinite limits applies. But it's the law that says $\lim_{x\to a} \frac{1}{(x-a)^k}$ is ∞ if k is even, but $-\infty$ from the left and $+\infty$ from the right if k is odd. In this case $k = 1$, which is odd, so the one-sided limits differ and consequently the two-sided limit that the question asks about does not exist.

How many vertical asymptotes does $\frac{1}{x^2-2x-3}$ have? Where are they?

Solution. From the factored form used to analyze the limit above, there will be two asymptotes, one at $x = -1$ and one at $x = 3$.

Use muPad to graph $\frac{1}{x^2-2x-3}$ and check that the graph is consistent with your answers to the previous two questions.

Solution. Use the muPad command `plot( 1 / (x^2 - 2*x - 3) )`. The result shows the two asymptotes, with the function heading towards different limits from different sides as it approaches each asymptote:



Problem 4. (From OpenStax, *Calculus Volume 1*, section 2.5.)

Prove that

$$\lim_{x\to 2} \frac{2x^2 - 3x - 2}{x - 2} = 5$$

Solution. Use an ϵ - δ proof, following the two-stage model used in class. The first stage infers a plausible δ from the condition $|f(x) - L| < \epsilon$. (For those interested in notation, The symbol $\longrightarrow$ that connects the steps in this derivation means implication, i.e., that the first condition being true implies that the second is true. But such implications don't have to work in reverse, which is why this style of proof needs its second stage.)

$$
\begin{aligned}
|f(x)-L|<\epsilon \quad &\longrightarrow \quad \left|\frac{2x^2-3x-2}{x-2}-5\right|<\epsilon \\
&\longrightarrow \quad \left|\frac{(2x+1)(x-2)}{x-2}-5\right|<\epsilon \\
&\longrightarrow \quad |(2x+1)-5|<\epsilon \\
&\longrightarrow \quad |2x-4|<\epsilon \\
&\longrightarrow \quad -\epsilon<2x-4<\epsilon \\
&\longrightarrow \quad -\frac{\epsilon}{2}<x-2<\frac{\epsilon}{2} \\
&\longrightarrow \quad |x-2|<\frac{\epsilon}{2} \\
&\longrightarrow \quad |x-a|<\frac{\epsilon}{2}
\end{aligned}
$$

Having found something that $|x-a|$ is less than, we guess that the "something" would make a good δ: $\delta = \frac{\epsilon}{2}$. The second stage of the proof now verifies that in fact any time $|x-a|<\delta$, $|f(x)-L|<\epsilon$:

$$
\begin{aligned}
|x-a|<\delta \quad &\longrightarrow \quad |x-2|<\frac{\epsilon}{2} \\
&\longrightarrow \quad -\frac{\epsilon}{2}<x-2<\frac{\epsilon}{2} \\
&\longrightarrow \quad -\epsilon<2(x-2)<\epsilon \\
&\longrightarrow \quad -\epsilon<2x-4<\epsilon \\
&\longrightarrow \quad |2x-4|<\epsilon \\
&\longrightarrow \quad |(2x+1)-5|<\epsilon \\
&\longrightarrow \quad \left|\frac{(2x+1)(x-2)}{x-2}-5\right|<\epsilon \\
&\longrightarrow \quad \left|\frac{2x^2-3x-2}{x-2}-5\right|<\epsilon \\
&\longrightarrow \quad |f(x)-L|<\epsilon
\end{aligned}
$$

Problem 5. One of the basic limit laws says that

$$\lim_{x\to a} c = c$$

for any constants a and c. Prove this law.

Solution. From the formal definition of limit, we need to show that for every positive ϵ, there is a δ that leads $|f(x)-L|$ to be less than ϵ whenever $0<|x-a|<\delta$. But notice that for this law, $f(x)=c$ and $L=c$, so $|f(x)-L|=|c-c|=0$. Since ϵ is positive, 0 is always less than ϵ, and so no matter what value δ has, $|f(x)-L|<\epsilon$. Therefore $\lim_{x\to a} c = c$.

Problem 6. Newton's law of gravitation says that the gravitational force pulling bodies of masses m_1 and m_2 (measured in kilograms) together is

$$F = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies in meters and G is a constant approximately equal to 6.67×10^{-11} $\mathrm{Nm}^2/\mathrm{kg}^2$. The force, F, is measured in newtons.

Conveniently, for purposes of this law symmetrical spherical masses can be treated as if all their mass is concentrated in a point at their center. This means, for example, that the force of gravity at the surface of a star or planet can be calculated using the star or planet's radius as r and its total mass as m_1.

Imagine that you, with some constant mass, are standing on the surface of a star[1] when it starts collapsing. The star collapses in such a way that its mass remains constant, but gets packed into an ever smaller sphere. Throughout this collapse, you keep standing on the shrinking surface of the star. Calculate your weight (i.e., the force of the star's gravity on you) in the limit as the star's radius approaches 0. To what extent do you think you can rely on this mathematical result as a model of physical reality? (For brownie points, but maybe not actual grade points, what is a star called if it collapses to the point that its radius approaches 0?)

Solution. Since G, m_1, and m_2 are all constants, we can simplify Newton's law for purposes of this problem to $F = \frac{K}{r^2}$ where K is the constant Gm_1m_2. Your weight as the star collapses is

$$\lim_{r \to 0} \frac{K}{r^2} = \infty$$

This is not a literal description of reality however (if there actually were a place where the force of gravity is infinite, that infinite gravity would be pulling us off Earth no matter how far away the source was). Rather, infinite results in mathematical models of the real world are taken as indications that the model has broken down — in this case, perhaps Newton's law doesn't apply to very dense bodies very close together, or maybe you can never get a non-zero mass into a truly zero volume, etc.

Stars that collapse toward zero radius are black holes.

[1] OK, there's a lot about standing on the surface of a star that needs a pretty active imagination...