Series Summary

A ***sequence*** is an ordered list of numbers: $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$, and a ***series*** is the sum of those numbers: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$. In either case, we want to determine if the sequence converges to a finite number or diverges and if the series converges to a finite number or diverges. If the series converges, that means that a sum of infinitely many numbers is equal to a finite number! If the sequence $\{a_n\}$ diverges or converges to anything other than 0, then the series $\sum a_n$ diverges. If the sequence $\{a_n\}$ converges to 0, then the series $\sum a_n$ may converge or may diverge.

For any given series $\sum a_n$ there are two associated sequences: the ***sequence of terms*** $\{a_n\}$ and the ***sequence of partial sums*** $\{s_n\}$, where $s_n = a_1 + a_2 + \ldots + a_n$. If $\sum a_n = L$, then $\lim_{n\to\infty} a_n = 0$ (as stated above) and $\lim_{n\to\infty} s_n = L$.

1 When can we calculate the sum of a series?

Unfortunately, we are unable to compute the exact sum of a series in most cases. However, there are a few examples that can be computed.

Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$

For $|r| < 1$, the series converges to $\frac{a}{1-r}$.

For $|r| \geq 1$, the series diverges.

Telescoping Series $\sum_{n=1}^{\infty} (b_n - b_{n+c})$

Also known as "canceling pairs", subsequent pairs of the series terms may cancel with each other.

2 Tests for determining if a series converges or diverges

In most cases, we will not be able to compute the exact sum of a series, but there are several tests which allow us to at least determine if a series is convergent or divergent. In some cases we can give approximations for the sum of a series as well.

Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

p-Series Test $\sum_{n=1}^{\infty} \frac{1}{n^p}$

For $p > 1$, the series converges.

For $p \leq 1$, the series diverges.

Integral Test

Applies when $a_n = f(n)$, and $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$.

The series $\sum_{n=1}^{\infty} a_n$ converges *if and only if* the integral $\int_1^{\infty} f(x)dx$ converges.

Comparison Test

Applies as long as a_n and b_n are always positive.

(i) If $a_n \leq b_n$ and $\sum b_n$ *converges*, then so does $\sum a_n$.

(ii) If $a_n \geq b_n$ and $\sum b_n$ *diverges*, then so does $\sum a_n$.

Limit Comparison Test

Applies as long as a_n and b_n are always positive, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive, finite number.

Then $\sum a_n$ converges *if and only if* $\sum b_n$ converges.

Alternating Series Test

$\sum(-1)^{n+1}a_n$

or $\sum(-1)^n a_n$

Applies when $a_n \geq 0$. The series converges if

(i) $a_n \geq a_{n+1}$, and

(ii) $\lim_{n\to\infty} a_n = 0$.

Absolute Convergence Test

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Ratio Test

Study this limit: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

(i) If the limit exists and is *less than* 1, the series $\sum a_n$ is *absolutely convergent* (and convergent).

(ii) If the limit exists and is *greater than* 1 (or if the limit diverges to infinity,) the series $\sum a_n$ *diverges.*

(iii) If the limit *equals* 1, the Ratio Test is useless.

Root Test

Study this limit: $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

(i) If the limit exists and is *less than* 1, the series $\sum a_n$ is *absolutely convergent* (and convergent).

(ii) If the limit exists and is *greater than* 1 (or if the limit diverges to infinity,) the series $\sum a_n$ *diverges.*

(iii) If the limit *equals* 1, the Root Test is useless.

3 How do we know which test to use?

1. If you can see easily that $\lim_{n\to\infty} a_n \neq 0$, apply the **Test for Divergence**.
2. Is $\sum a_n$ a p**-series** or **geometric series**? If yes, apply those tests.
3. If the series $\sum a_n$ appears to be a **telescoping sum**, then find a closed formula for the partial sum s_n and use $\sum a_n = \lim_{n\to\infty} s_n$.
4. Is $\sum a_n$ *similar to* a p-series or geometric series? If yes, try one of the Comparison Tests. You will compare $\sum a_n$ with the series $\sum b_n$ that $\sum a_n$ is similar to.
   (a) If the inequality works out the way you need it to, you will you the **Comparison Test**.
   (b) If the inequality does not work out the way you need it to, try the **Limit Comparison Test**.
5. If $a_n = f(n)$ and $\int_1^\infty f(x)\,dx$ is easily evaluated, use the **Integral Test**.
6. If the series is of the form $\sum(-1)^{n+1}a_n$ or $\sum(-1)^n a_n$, try the **Alternating Series Test**.
7. When earlier tests can not be used simply because some of the terms may be negative, try using the **Absolute Convergence Test**.
8. Series involving factorials (e.g. $n!$) or n^{th} powers of a constant (e.g. 4^n) can often be studied with the **Ratio Test** (if an easier test does not work.)
9. When a_n looks like $(\cdots)^n$, and the term inside the parentheses *also* involves n, try the **Root Test**.