Theorem 1: The Onto and One-to-One Theorem

Let A be the $m \times p$ standard matrix for the linear transformation $T : \mathbb{R}^p \to \mathbb{R}^m$. Then the following statements are equivalent:

1. T is both one-to-one and onto,

2. $T(\vec{x}) = \vec{b}$ has exactly one solution $\vec{x} \in \mathbb{R}^p$ for every $\vec{b} \in \mathbb{R}^m$,

3. $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} \in \mathbb{R}^p$ for every $\vec{b} \in \mathbb{R}^m$, and

4. REF(A) has a pivot in <u>every row</u> and <u>every column</u>.

Theorem 2: The Invertible Matrix Theorem

Let A be the $n \times n$ standard matrix for the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then the following statements are equivalent:

1. A is invertible, Caution: A **must** be a **square** matrix.
2. $\text{RREF}(A) = I_n$,
3. $\text{REF}(A)$ has a pivot in <u>every row</u> and <u>every column</u>,
4. $\text{REF}(A)$ has n pivots,
5. $A\vec{x} = \vec{0}$ has exactly one solution $\vec{x} = \vec{0}$,
6. $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} = A^{-1}\vec{b}$ for every $\vec{b} \in \mathbb{R}^n$,
7. The columns (and rows) of A are linearly independent,
8. The columns (and rows) of A span $\mathbb{R}^n$,
9. T is one-to-one,
10. T is onto, and
11. $T(\vec{x}) = \vec{b}$ has exactly one solution $\vec{x} \in \mathbb{R}^p$ for every $\vec{b} \in \mathbb{R}^m$.