Theorem F: The Onto Theorem

Let A be the $m \times p$ standard matrix for the linear transformation $T : \mathbb{R}^p \to \mathbb{R}^m$.

1. T is not onto if $m > p$ (i.e. if A has more rows than columns).
2. T is onto (and $m \leq p$) if and only if:
   (a) $A\vec{x} = \vec{b}$ always has at least one solution for any $\vec{b} \in \mathbb{R}^m$,
   (b) Any $\vec{b} \in \mathbb{R}^m$ is a linear combination of columns of A,
   (c) Columns of A span (or generate) all of $\mathbb{R}^m$,
   (d) REF(A) has a pivot in every row,
   (e) REF(A) has m pivot columns and $p - m$ free columns,
   (f) Of the p columns of A, m of them are linearly independent, and
   (g) All rows of A are linearly independent.

Theorem G: The One-to-One Theorem

Let A be the $m \times p$ standard matrix for the linear transformation $T : \mathbb{R}^p \to \mathbb{R}^m$.

1. T is <u>not one-to-one</u> if $p > m$ (i.e. if A has more columns than rows).
2. T is <u>one-to-one</u> (and $p \leq m$) if and only if:
   (a) $\vec{x} \neq \vec{y} \quad \Rightarrow \quad T(\vec{x}) \neq T(\vec{y})$
   (b) $T(\vec{x}) = T(\vec{y}) \quad \Rightarrow \quad \vec{x} = \vec{y}$
   (c) $A\vec{x} = \vec{b}$ has at most one solution for any $\vec{b} \in \mathbb{R}^m$,
   (d) $T(\vec{x}) = \vec{0}$ has only one solution $\vec{x} = \vec{0}$,
   (e) $A\vec{x} = \vec{0}$ has only one solution $\vec{x} = \vec{0}$,
   (f) Rows of A span (or generate) all of $\mathbb{R}^p$,
   (g) REF(A) has a pivot in <u>every column</u>,
   (h) Of the m rows of A, p of them are linearly independent, and
   (i) All columns of A are linearly independent.

Theorem 1: The Onto and One-to-One Theorem

Let A be the $m \times p$ standard matrix for the linear transformation $T : \mathbb{R}^p \to \mathbb{R}^m$. Then the following statements are equivalent:

1. $T(\vec{x}) = \vec{b}$ has exactly one solution $\vec{x} \in \mathbb{R}^p$ for every $\vec{b} \in \mathbb{R}^m$,
2. A$A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} \in \mathbb{R}^p$ for every $\vec{b} \in \mathbb{R}^m$,
3. T is both one-to-one and onto, and
4. REF(A) has a pivot in <u>every row</u> and <u>every column</u>.