1 Equivalence Relation

Definition 1. *An* ***equivalence relation*** *is a relationship on a set, generally denoted by "$\sim$", that is reflexive, symmetric, and transitive for everything in the set.*

1. *(Reflexivity)* $a \sim a$*,*
2. *(Symmetry) if* $a \sim b$ *then* $b \sim a$*,*
3. *(Transitivity) if* $a \sim b$ *and* $b \sim c$ *then* $a \sim c$*.*

Equivalence relations are often used to group together objects that are similar, or "equivalent", in some sense.

2 Examples

Example: The relation "is equal to", denoted "=", is an equivalence relation on the set of real numbers since for any $x, y, z \in \mathbb{R}$:

1. (Reflexivity) $x = x$,
2. (Symmetry) if $x = y$ then $y = x$,
3. (Transitivity) if $x = y$ and $y = z$ then $x = z$.

All of these are true.

Example: Let "$\simeq$" denote the relation on the set of symmetric matrices (recall symmetric means $A = A^t$) defined as follows. $A \simeq B$ if $A = B^t$. This is an equivalence relation since for any symmetric matrices A, B, C:

1. (Reflexivity) $A \simeq A$ since $A = A^t$.
2. (Symmetry) If $A \simeq B$ then $A = B^t$, but then $B = (B^t)^t = (A)^t = A^t$. And this implies $B \simeq A$.
3. (Transitivity) If $A \simeq B$ and $B \simeq C$ then $A = B^t$ and $B = C^t$. Of course, since these are symmetric matrices, we know $B = B^t$. Thus $A = B^t = B = C^t$, and therefore $A \simeq C$.

(Of course this is actually a "stupid" equivalence relation since it is the same as the "is equal to" equivalence relation on the set of symmetric matrices. Do you see why?)

Non-example: The relation "is less than or equal to", denoted "$\leq$", is NOT an equivalence relation on the set of real numbers. For any $x, y, z \in \mathbb{R}$, "$\leq$" is reflexive and transitive but NOT necessarily symmetric.

1. (Reflexivity) Of course $x \leq x$ is true since $x = x$.
2. (Symmetry) If $x \leq y$ then it is not necessarily true that $y \leq x$. For example, $5 \leq 7$, but $7 \not\leq 5$.
3. (Transitivity) If $x \leq y$ and $y \leq z$ then $x \leq z$ since $x \leq y \leq z$.