1 Definitions

1. (0 points) Let $U$ be a subset of a vector space $V$. Let $S = \{v_1, v_2, \ldots, v_n\}$ be another subset of $V$.
   (a) Define “$U$ is a subspace of $V$”.
   (b) Define “$S$ is linearly independent”.
   (c) Define “$S$ generates $V$”.

2 Vector Spaces and Subspaces

2. (0 points)
   (a) Give three examples of 4-dimensional vector spaces.
   (b) Give one example of an infinite dimensional vector space.
   (c) Give an example of a zero-dimensional vector space.

3. (0 points) Let $S_1$ and $S_2$ be subspaces of a vector space $V$. Prove that the union $S_1 \cup S_2$ is a subspace of $V$ if and only if one is contained in the other (that is, either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$).

Solution: $(\iff)$ $S_1$ and $S_2$ are subspaces. If $S_1 \subseteq S_2$, then $S_1 \cup S_2 = S_2$ is a subspace. If $S_2 \subseteq S_1$, then $S_1 \cup S_2 = S_1$ is a subspace. We’ve proved one direction.

$(\implies)$ $S_1$ and $S_2$ are subspaces, and suppose $S_1 \cup S_2$ is a subspace. If $S_1 \subseteq S_2$, then we are done. If $S_1 \not\subseteq S_2$, then we need to show $S_2 \subseteq S_1$.

Choose $x \in S_2$. Since $S_1 \not\subseteq S_2$ there must be some vector in $S_1$ that is not in $S_2$, call it $y$. So $y \in S_1$, but $y \notin S_2$. Since $S_1 \cup S_2$ is a subspace, it is closed under addition and
\[ x + y \text{ must be in } S_1 \cup S_2 \text{ since } x \in S_2 \subset S_1 \cup S_2 \text{ and } y \in S_1 \subset S_1 \cup S_2. \] Thus we must have either \( x + y \in S_1 \) or \( x + y \in S_2 \).

If \( x + y \in S_2 \), then since \( x \in S_2 \) and \( S_2 \) is a subspace (i.e. closed under the operations) we have \( y = (x + y) - x \in S_2 \), which contradicts the fact that \( y \notin S_2 \). Thus \( x + y \in S_1 \). However, since \( y \in S_1 \) and \( S_1 \) is a subspace (i.e. closed under the operations) we have \( x = (x + y) - y \in S_1 \). Therefore, \( S_2 \subset S_1 \).

\section*{3 Linear Independence, Generating Sets, and Bases}

\textbf{4. (0 points)} Let \( S = \{x^2 + 3x, x - 2\} \) be a subset of \( P_2(\mathbb{R}) \).

(a) Explain why \( S \) is \textit{not} a basis of \( P_2(\mathbb{R}) \).

(b) Is \( \frac{1}{2}x^2 + 2 \) in \( \text{span}(S) \)? Explain.

(c) Is \( 2x^2 + 5x + 4 \) in \( \text{span}(S) \)? Explain.

\textbf{5. (0 points)} Consider the 3 vectors in \( \mathbb{R}^3 \) given by \( v_1 = (1, 1, -1), v_2 = (1, 1, 1) \), and \( v_3 = (3, 5, 7) \). Decide whether these 3 vectors provide a basis for \( \mathbb{R}^3 \). Justify your answer.

\textbf{6. (0 points)} Let \( W \) be the subspace of \( \mathbb{R}^3 \) given by

\[ W = \{(x, y, z) \mid x + y + z = 0 \text{ and } x - y - z = 0\}. \]

Find a basis for \( W \) and the dimension of \( W \).

\textbf{7. (0 points)} Let \( S = \{v_1, v_2, \ldots, v_n\} \) be a set of \( n \) vectors in a vector space \( V \). Show that if \( S \) is linearly independent and the dimension of \( V \) is \( n \), then \( S \) is a basis of \( V \).

\textbf{Solution:} This is Corollary 2 (b) at the top of page 48 of the textbook. The proof is found there.
8. (0 points) Consider the subset \( S = \{ x^3 - 2x^2 + 1, 4x^2 - x + 3, 3x - 2 \} \) of \( P_3(\mathbb{R}) \).

(a) Explain how you know that \( S \) does not generate \( P_3(\mathbb{R}) \).

**Solution:** Since \( S \) has 3 vectors and the dimension of \( P_3(\mathbb{R}) \) is 4, \( S \) cannot generate \( P_3(\mathbb{R}) \).

(b) Can you add a vector \( v \) to \( S \) so that \( S \cup \{ v \} \) is a basis of \( P_3(\mathbb{R}) \)? Justify and find such a vector if possible.

**Solution:** As long as \( S \) is linearly independent we know that \( S \) can be extended to a basis. To see \( S \) is linearly independent suppose that
\[
a(x^3 - 2x^2 + 1) + b(4x^2 - x + 3) + c(3x - 2) = 0.
\]
This clearly implies that \( a = 0 \) since only one term has an \( x^3 \). So now
\[
b(4x^2 - x + 3) + c(3x - 2) = 0,
\]
and again we see that \( b = 0 \). Clearly \( c \) must also be 0. Furthermore, we can add \( v = 1 \) as the last vector using a similar argument to show this new set is linearly independent. Then since the dimension of \( P_3(\mathbb{R}) \) is 4, we know this new set is a basis.

9. (0 points) Let \( V \) be a vector space over \( \mathbb{R} \), and let \( x, y, z \in V \). Prove that \( \{ x, y, z \} \) is linearly independent if and only if \( \{ x + y, y + z, z + x \} \) is linearly independent.

**Solution:** \( \implies \) Assume that \( \{ x, y, z \} \) is linearly independent. Suppose there are \( a, b, c \in \mathbb{R} \) such that
\[
a(x + y) + b(y + z) + c(z + x) = 0.
\]
So \( 0 = a(x + y) + b(y + z) + c(z + x) = (a + c)x + (a + b)y + (b + c)z \), and this means that \( a + c = a + b = b + c = 0 \) since \( \{ x, y, z \} \) is linearly independent. Clearly from those equalities we have \( a = b = c = 0 \). Therefore, \( \{ x + y, y + z, z + x \} \) is also linearly independent.

\( \impliedby \) Assume that \( \{ x + y, y + z, z + x \} \) is linearly independent. Suppose there are \( a, b, c \in \mathbb{R} \) such that
\[
a x + b y + c z = 0.
\]
So \(0 = ax + by + cz = \left(\frac{a+b-c}{2}\right)(x + y) + \left(\frac{b+c-a}{2}\right)(y + z) + \left(\frac{c+a-b}{2}\right)(z + x)\), and this means that \(a + b - c = b + c - a = c + a - b = 0\) since \(\{x + y, y + z, z + x\}\) is linearly independent. Clearly from those equalities we have \(a = b = c = 0\). Therefore, \(\{x, y, z\}\) is also linearly independent.

10. (0 points) Let \(S_1\) and \(S_2\) be subsets of a vector space \(V\) over a field \(F\). Prove that

\[
\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2).
\]

**Solution:** Let \(x \in \text{span}(S_1 \cap S_2)\). Then there exist vectors \(v_1, v_2, \ldots, v_n \in S_1 \cap S_2\) and coefficients \(a_1, a_2, \ldots, a_n \in F\) such that \(x = a_1v_1 + a_2v_2 + \cdots + a_nv_n\). But since \(v_1, v_2, \ldots, v_n \in S_1\), we see that \(x \in \text{span}(S_1)\). Similarly, since \(v_1, v_2, \ldots, v_n \in S_2\), we see that \(x \in \text{span}(S_2)\). Thus we have \(x \in \text{span}(S_1) \cap \text{span}(S_2)\).

11. (0 points) Consider the vector space \(V = P_1(\mathbb{R})\).

(a) Explain why you know that the set \(\beta = \{1 + x, 1 - 2x\}\) is a basis of \(V\).

**Solution:** Since neither vector is a multiple of the other, \(\beta\) is linearly independent. Since the dimension of \(V\) is 2 and \(\beta\) has 2 elements, it must be a basis.

(b) Express \(p(x) = 2x - 3\) as a linear combination of \(\beta\).

**Solution:** \(p(x) = 2x - 3 = (-4/3)(1 + x) + (-5/3)(1 - 2x)\)