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Bordism Invariants of the Mapping Class Group

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ABSTRACT

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We define new bordism and spin bordism invariants of certain subgroups of the mapping class group of a surface. In particular, they are invariants of the Johnson filtration of the mapping class group. The second and third terms of this filtration are the well-known Torelli group and Johnson subgroup, respectively. We introduce a new representation in terms of spin bordism, and we prove that this single representation contains all of the information given by the Johnson homomorphisms, the Birman-Craggs homomorphisms, and the Morita homomorphisms.

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Contents

List of Figures

Chapter 1

Introduction

1.1 Background

Let $\Sigma_{g,1}$ be a compact, oriented surface of genus g with one boundary component. Let $\Gamma_{g,1}$ be the mapping class group of $\Sigma_{g,1}$. That is, $\Gamma_{g,1}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,1}$ which fix the boundary. The study of mapping class groups has important applications in many different areas of topology, differential geometry, and algebraic geometry. Here we are particularly interested in $\Gamma_{g,1}$ within the area of 3-manifold topology.

The mapping class group $\Gamma_{g,1}$ acts naturally by automorphisms on the fundamental group $F = \pi_1(\Sigma_{g,1})$, which is a free group of rank $2g$. Then we have the induced representation $\Gamma_{g,1} \to \mathrm{Aut}(F)$, and this representation is known classically to be injective. Let $\{F_k\}_{k\geq 1}$ be the lower central series of F. That is, $F_1 = F$ and the rest of the terms are defined inductively by $F_{k+1} = [F_k, F_1]$ for any $k \geq 1$. Then $\Gamma_{g,1}$ acts naturally on the nilpotent quotients F/F_k, providing a series of representations

$$\rho_k : \Gamma_{g,1} \to \mathrm{Aut}\left(\frac{F}{F_k}\right).$$

Note that F/F_2 is isomorphic to the first homology group $H_1 = H_1(\Sigma_{g,1}; \mathbb{Z})$, and ρ_2 is the same as the classical representation $\Gamma_{g,1} \to \mathrm{Sp}(2g;\mathbb{Z})$ of the mapping class group onto the Siegel modular group, which is the group of symplectic automorphisms of H_1 with respect to the skew-symmetric intersection pairing.

The *generalized Johnson subgroup* $\mathcal{J}(k) \subseteq \Gamma_{g,1}$ is defined to be the kernel of ρ_k. That is, $\mathcal{J}(k)$ is the subgroup of the mapping class group consisting of those homeomorphisms which induce the identity on F/F_k. The subgroup $\mathcal{J}(2) = \mathcal{T}_{g,1}$ is more commonly known as the *Torelli group*, and $\mathcal{J}(3) = \mathcal{K}_{g,1}$ is traditionally referred to as the *Johnson subgroup*. The Johnson subgroup was originally defined to be the subgroup of $\Gamma_{g,1}$ generated by all Dehn twists about separating simple closed curves on $\Sigma_{g,1}$. The fact that these two definitions of $\mathcal{K}_{g,1}$ are equivalent was proved by D. Johnson in [J5].

The structure of the subgroup $\mathcal{J}(k)$ is far from being completely understood. In the case of $g = 2$, G. Mess proved in [Me] that $\mathcal{J}(2) = \mathcal{J}(3)$ and is an infinitely generated free group. For $g \geq 3$, Johnson showed in [J4] that the Torelli group $\mathcal{J}(2)$ is finitely generated. However, it is still not known whether or not $\mathcal{J}(2)$ has a finite presentation for $g \geq 3$. It is also not known if $\mathcal{J}(k)$ is finitely generated for $k \geq 3$. And while $\mathcal{J}(2)$ and $\mathcal{J}(3)$ have very nice generating sets, there is no known generating set for $\mathcal{J}(k)$ for $k \geq 4$.

To get a better understanding of the structure of the subgroup $\mathcal{J}(k)$, it is natural to seek good abelian representations for it. That is, we would hope to understand

$\mathcal{J}(k)$ better by investigating abelian quotients of it. The first such quotient of the Torelli group $\mathcal{J}(2)$ was given by a homomorphism due to D. Sullivan in [Su]. Johnson gave another homomorphism for $\mathcal{J}(2)$, of which Sullivan's is a quotient, in [J2]. He later generalized this homomorphism to $\mathcal{J}(k)$ for all $k \geq 2$ in [J3], thus giving a family of homomorphisms

$$\tau_k : \mathcal{J}(k) \to \mathrm{Hom}\left(H_1, \frac{F_k}{F_{k+1}}\right),$$

now known as the *Johnson homomorphisms.* In the case $k = 2$, the image of τ_2 is known to be a submodule $D_2(H_1)$ of $\mathrm{Hom}(H_1, F/F_2) = \mathrm{Hom}(H_1, H_1)$. Moreover, the kernel of τ_2 is known to be $\mathcal{J}(3)$. In general, $\ker \tau_k = \mathcal{J}(k+1)$. However, the image of τ_k is not known for $k \geq 3$, and it is a fundamental problem in the study of the mapping class group to determine its image.

In [BC] J. Birman and R. Craggs produced a collection of abelian quotients of $\mathcal{J}(2)$ given by homomorphisms onto $\mathbb{Z}_2$

$$\rho : \mathcal{J}(2) \to \mathbb{Z}_2.$$

These are finite in number and unrelated to Johnson's homomorphism. However, Johnson showed in [J6] that the Johnson homomorphism τ_2 and the totality of these *Birman-Craggs homomorphisms*, together, completely determine the abelianization of the Torelli group $\mathcal{J}(2)$ for $g \geq 3$. The abelianization of $\mathcal{J}(k)$ is not known for $k > 2$.

1.2 Summary of Results

For any mapping class $f \in \Gamma_{g,1}$, there is an associated compact, oriented 3-manifold $T_{f,1}$ known as the mapping torus of f. That is, $T_{f,1}$ is $\Sigma_{g,1} \times [0,1]$ with $x \times \{0\}$ glued to $f(x) \times \{1\}$. This association allows us to relate the mapping class group to various 3-dimensional bordism groups and develop interesting invariants of $\Gamma_{g,1}$. Construct from $T_{f,1}$ a closed 3-manifold T_f^γ by filling in the boundary $\partial T_{f,1} = \partial\Sigma_{g,1} \times S^1$ with the solid torus $\partial\Sigma_{g,1} \times D^2$. If $f \in \mathcal{J}(k)$ there is a canonical epimorphism

$$\pi_1(T_f^\gamma) \twoheadrightarrow \frac{\pi_1(T_f^\gamma)}{\left(\pi_1(T_f^\gamma)\right)_k} \cong \frac{F}{F_k}.$$

Let $\phi_{f,k}^\gamma : T_f^\gamma \to K(F/F_k, 1)$ be a continuous map inducing this epimorphism, where $K(F/F_k, 1)$ is an Eilenberg-MacLane space.

Let $\Omega_3(F/F_k)$ be the 3-dimensional oriented bordism group over F/F_k. An element of this group is a bordism class (M, ϕ) consisting of a closed, oriented 3-manifold M and a continuous map $\phi : M \to K(F/F_k, 1)$. Two elements (M_0, ϕ_0) and (M_1, ϕ_1) of $\Omega_3(F/F_k)$ are equivalent, or bordant, if there exists a 4-manifold W whose boundary is $\partial W = M_0 \amalg -M_1$ and a continuous map $\Phi : W \to K(F/F_k, 1)$ such that $\Phi|_{M_i} = \phi_i$. We may then ask when $\left(T_f^\gamma, \phi_{f,k}^\gamma\right)$ is a trivial bordism class in $\Omega_3(F/F_k)$.

Theorem 5.3 $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \in \Omega_3(F/F_k)$ *is trivial if and only if* $f \in \mathcal{J}(2k-1)$.

This allows us to define a new faithful abelian representation of $\mathcal{J}(k)/\mathcal{J}(2k-1)$.

Theorem 5.2 *The map*

$$\sigma_k : \mathcal{J}(k) \to \Omega_3\left(\frac{F}{F_k}\right)$$

defined by $\sigma_k(f) = \left(T_f^\gamma, \phi_{f,k}^\gamma\right)$ *is a well-defined homomorphism.*

Corollary 5.4 *The kernel of the homomorphism σ_k is $\mathcal{J}(2k-1)$.*

This representation contains all of the information given by the Johnson homomorphism $\tau_k : \mathcal{J}(k) \to \mathrm{Hom}(H_1, F_k/F_{k+1})$, however, the kernel of σ_k is much finer than the kernel of τ_k, which is $\mathcal{J}(k+1)$. Moreover, the quotient $\mathcal{J}(k)/\mathcal{J}(2k-1)$ is much closer to the abelianization of $\mathcal{J}(k)$ than $\mathcal{J}(k)/\mathcal{J}(k+1)$ is if $k > 2$. To achieve a faithful representation of the quotient $\mathcal{J}(k)/\mathcal{J}(2k-1)$, one must either consider all of the homomorphisms $\{\tau_k, ..., \tau_{2k-2}\}$ or simply consider σ_k alone.

Another representation is given in terms of spin bordism. The spin bordism group $\Omega_3^{spin}(F/F_k)$ is defined the same way as $\Omega_3(F/F_k)$ except that elements are triples (M, ϕ, s), where s is a spin structure on M, and the 4-manifold W is required to have a spin structure that induces the given spin structure on its boundary components.

Theorem 6.1 *Fix a spin structure on $\Sigma_{g,1}$. Let σ denote the induced spin structure on T_f^γ for $f \in \mathcal{J}(k)$, $k \geq 2$. Then there is a well-defined homomorphism*

$$\eta_k : \mathcal{J}(k) \to \Omega_3^{spin}\left(\frac{F}{F_k}\right)$$

defined by $\eta_k(f) = \left(T_f^\gamma, \phi_{f,k}^\gamma, \sigma\right)$.

This second representation contains, in addition to the information given by the Johnson homomorphism, all of the information given by the Birman-Craggs homomorphisms $\rho_\sigma : \mathcal{J}(2) \to \mathbb{Z}_2$. In particular, the homomorphism η_2 has the desirable advantage of incorporating all of the information of the Johnson and Birman-Craggs homomorphisms into a single homomorphism. Moreover, η_k gives evidence of the existence of more new invariants.

1.3 Outline of Thesis

In Chapters 2, 3, and 4 we review some of the well-known representations of the mapping class group $\Gamma_{g,1}$. All of the information given in these chapters is documented elsewhere, so those who are familiar with this information may choose to skip these chapters. Chapter 2 reviews the Johnson homomorphism, Massey products, and a refinement of the Johnson homomorphism due to Morita. Chapter 3 reviews the Birman-Craggs homomorphism, and Chapter 4 discusses the abelianization of the Torelli group, which is completely determined by the Johnson homomorphism and the Birman-Craggs homomorphism.

In Chapter 5 we define a new representation of the mapping class group and give bordism invariants of the generalized Johnson subgroups. We consider the 3-dimension oriented bordism group over F/F_k, where F is the fundamental group of $\Sigma_{g,1}$ and F_k denotes the lower central series of F. We also investigate the relationship between this new representation and the Johnson homomorphism.

In Chapter 6 we define another new representation of the mapping class group in terms of spin bordism. In this case we consider the 3-dimension spin bordism group over F/F_k. We give the relationship between these spin bordism invariants and the oriented bordism invariants and also the relationship between this spin bordism representation and the Birman-Craggs homomorphism. We then investigate the possibility of more new invariants.

Chapter 2

The Johnson Homomorphism

2.1 Johnson's Original Definition of τ_k

In this section we give a description of Johnson's homomorphisms. Let $\Sigma_{g,1}$ be a compact, oriented surface of genus g with one boundary component. Then the fundamental group $F = \pi_1(\Sigma_{g,1})$ is a free group with $2g$ generators, and $\{F_k\}_{k\geq 1}$ is the lower central series of F. That is, $F_1 = F$ and $F_{k+1} = [F_k, F_1]$ for any $k \geq 1$. Let the *generalized Johnson subgroup* $\mathcal{J}(k)$ be the subgroup of the mapping class group consisting of those homeomorphisms that induce the identity on F/F_k.

Consider any $f \in \mathcal{J}(k)$. Choose a representative $\gamma \in \pi_1(\Sigma_{g,1}) = F$ for any given element $[\gamma] \in H_1 = H_1(\Sigma_{g,1}; \mathbb{Z}) = F/F_2$, and consider the element $f_*(\gamma)\gamma^{-1}$ which belongs to F_k since $f \in \mathcal{J}(k)$ implies f_* acts trivially on F/F_k. Then let $[f_*(\gamma)\gamma^{-1}] \in F_k/F_{k+1}$ denote the equivalence class of $f_*(\gamma)\gamma^{-1}$ under the projection $F_k \to F_k/F_{k+1}$. Then we define the *Johnson homomorphisms*

$$\tau_k : \mathcal{J}(k) \to \mathrm{Hom}\left(H_1, \frac{F_k}{F_{k+1}}\right)$$

by letting $\tau_k(f)$ be the homomorphism $[\gamma] \to [f_*(\gamma)\gamma^{-1}]$. The skew-symmetric intersection pairing on H_1 defines a canonical isomorphism $H_1 \cong \mathrm{Hom}(H_1, \mathbb{Z})$, and this induces an isomorphism

$$\mathrm{Hom}\left(H_1, \frac{F_k}{F_{k+1}}\right) \cong \mathrm{Hom}(H_1, \mathbb{Z}) \otimes \frac{F_k}{F_{k+1}} \cong H_1 \otimes \frac{F_k}{F_{k+1}}.$$

Thus we could also write

$$\tau_k : \mathcal{J}(k) \to H_1 \otimes \frac{F_k}{F_{k+1}}.$$

This is Johnson's original definition [J3], but there are several equivalent definitions of his homomorphism. Also in [J3], one can see a definition in terms of the intersection ring of the mapping torus of f. There is a definition of τ_k in terms of the Magnus representation of the mapping class group $\Gamma_{g,1}$ that may be found in [Ki] or [M1]. One can also view τ_k as the restriction of the representation

$$\rho_{k+1} : \Gamma_{g,1} \to \mathrm{Aut}\left(\frac{F}{F_{k+1}}\right)$$

to $\mathcal{J}(k) = \ker \rho_k$ via the extension

$$0 \to \mathrm{Hom}\left(H_1, \frac{F_k}{F_{k+1}}\right) \to \mathrm{Aut}\left(\frac{F}{F_{k+1}}\right) \to \mathrm{Aut}\left(\frac{F}{F_k}\right) \to 1.$$

For a detailed description of this viewpoint, the reader is encouraged to see the work of Morita in [M1] or [M2]. The final definition we mention in this paper will be given in Section 2.3, and it was stated by Johnson in [J3] and verified by Kitano in [Ki]. This definition gives a computable description of τ_k in terms of Massey products of mapping tori.

We complete this section with a few well-known facts about the Johnson homomorphisms τ_k and the subgroups $\mathcal{J}(k)$. It was shown by Morita in [M1] that

$$[\mathcal{J}(k), \mathcal{J}(l)] \subset \mathcal{J}(k+l-1).$$

In particular, the commutator subgroup

$$[\mathcal{J}(k), \mathcal{J}(k)] \subset \mathcal{J}(2k-1) \subset \mathcal{J}(k+1)$$

for $k \geq 2$. As mentioned in the introduction, $\ker \tau_k = \mathcal{J}(k+1)$. Then the image of τ_k is isomorphic to the abelian quotient $\mathcal{J}(k)/\mathcal{J}(k+1)$. Thus the information provided by the $k-1$ homomorphisms $\tau_k, ..., \tau_{2k-2}$ can be combined to determine the abelian quotient $\mathcal{J}(k)/\mathcal{J}(2k-1)$. Unfortunately this only at most detects the free-abelian part of the abelianization $\mathcal{J}(k)/\left[\mathcal{J}(k), \mathcal{J}(k)\right] \cong H_1(\mathcal{J}(k))$. For example, the image of τ_2 is given by $\mathcal{J}(2)/\mathcal{J}(3) = \mathcal{T}_{g,1}/\mathcal{K}_{g,1}$, and $\mathcal{J}(2)/\mathcal{J}(3) \otimes \mathbb{Q} \cong H_1(\mathcal{T}_{g,1}; \mathbb{Q})$, whereas the abelianization of the Torelli group $H_1(\mathcal{T}_{g,1})$ has 2-torsion. We will discuss this 2-torsion in more detail in Chapter 3.

2.2 Massey Products

Let (X, A) be a pair of topological spaces, and unless otherwise stated we assume that the coefficients for homology and cohomology groups are always the integers $\mathbb{Z}$. In this section we will give the definition of the Massey product

$$H^1(X, A) \otimes \cdots \otimes H^1(X, A) \to H^2(X, A)$$

since these are the only dimensions that we are interested in using, and we will give a few useful properties of which we wish to take advantage. The general definition is completely analogous except for various sign conventions, and we refer the reader to D. Kraines [Kr]. For a more complete description of this specific definition we are giving and for some useful examples, we refer you to R. Fenn's book [Fe].

Massey products may be viewed as higher order analogues of cup products and are defined when certain cup products vanish. Let $u_1, ..., u_n \in H^1(X, A)$ be cohomology classes with cocycle representatives $a_1, ..., a_n \in C^1(X, A)$, respectively. A *defining set* for the Massey product $\langle u_1, ..., u_n \rangle$ is a collection of cochains $a = (a_{i,j})$, $1 \leq i \leq j \leq n$ and $(i, j) \neq (1, n)$, satisfying

(1) $a_{i,i} = a_i$ for any $i \in \{1, ..., n\}$,

(2) $a_{i,j} \in C^1(X, A)$,

(3) $\delta a_{i,j} = \sum_{r=i}^{j-1} a_{i,r} \cup a_{r+1,j}$.

For such a defining set a consider the cocycle $u(a) \in C^2(X, A)$ given by

$$u(a) = \sum_{r=1}^{n-1} a_{1,r} \cup a_{r+1,n}.$$

The *Massey product* $\langle u_1, ..., u_n \rangle$ is defined if a defining set a exists, and it is defined to be the subset of $H^2(X, A)$ consisting of the values $u(a)$ of all such defining sets a.

The length 1 Massey product $\langle u_1 \rangle$ is simply defined to be u_1, and its defining set is any cocycle representative of u_1. The length 2 Massey product $\langle u_1, u_2 \rangle$ is the cup product $u_1 \cup u_2$. The triple Massey product $\langle u_1, u_2, u_3 \rangle$ is defined only when $\langle u_1, u_2 \rangle$ and $\langle u_2, u_3 \rangle$ are zero. As you may notice from the definition, Massey products of length 3 or greater may not be uniquely defined but in fact may be a set of elements. However, if a sufficient number of smaller Massey products vanish, then $\langle u_1, ..., u_n \rangle$ is uniquely defined. We have the following useful properties.

(2.2.1) *Uniqueness.* For $n \geq 3$, the Massey product $\langle u_1, ..., u_n \rangle$ is uniquely defined if all Massey products of length less than n are defined and vanish. (This hypothesis is stronger than necessary for uniqueness, but it is sufficient for us. See Fenn [Fe] for details.)

(2.2.2) *Naturality.* Let (Y, B) be a pair of topological spaces, and consider a map of pairs $f : (Y, B) \to (X, A)$. If $\langle u_1, ..., u_n \rangle$ is defined then so is $\langle f^*(u_1), ..., f^*(u_n) \rangle$, and $f^* \langle u_1, ..., u_n \rangle \subset \langle f^*(u_1), ..., f^*(u_n) \rangle$. Furthermore, if f^* is an isomorphism, then equality holds.

2.3 Massey Product Description of τ_k

We are now prepared to describe Johnson's homomorphisms τ_k using Massey products of mapping tori. For a more complete description, see the work of T. Kitano [Ki]. As before, $\Sigma_{g,1}$ is an oriented surface of genus g with one boundary component $\partial\Sigma_{g,1}$. Consider any homeomorphism $f \in \mathcal{J}(k)$, and let $T_{f,1}$ denote the mapping torus of f. That is, $T_{f,1}$ is $\Sigma_{g,1} \times [0, 1]$ with $x \times \{0\}$ glued to $f(x) \times \{1\}$. Note that the boundary $\partial T_{f,1}$ is the torus $\partial\Sigma_{g,1} \times S^1$. With the natural orientation on $[0, 1]$, we have a local orientation on $T_{f,1}$ given by the product orientation. Moreover, since $f \in \mathcal{J}(k)$ acts trivially on $H_1 = H_1(\Sigma_{g,1})$ as long as $k \geq 2$, the mapping torus $T_{f,1}$ is an oriented homology $\Sigma_{g,1} \times S^1$, but the Massey product structure may be different than that of $\Sigma_{g,1} \times S^1$.

First, fix a basis $\{\alpha_1, ..., \alpha_{2g}\}$ for the free group $F = \pi_1(\Sigma_{g,1})$. Then if γ represents a generator of $\pi_1(S^1)$, we get the following presentation of $\pi_1(T_{f,1})$:

$$\pi_1(T_{f,1}) = \left\langle \alpha_1, ..., \alpha_{2g}, \gamma \mid [\alpha_1, \gamma]\, f_*(\alpha_1)\alpha_1^{-1}, ..., [\alpha_{2g}, \gamma]\, f_*(\alpha_{2g})\alpha_{2g}^{-1} \right\rangle .$$

By denoting the homology classes of α_i and γ by x_i and y, respectively, we obtain a basis for $H_1(T_{f,1})$:

$$\{x_1, ..., x_{2g}, y\} \in H_1(T_{f,1}).$$

Then since $H^1(T_{f,1}) \cong \mathrm{Hom}(H_1(T_{f,1}), \mathbb{Z})$, we have a dual basis for $H^1(T_{f,1})$:

$$\left\{x_1^*, ..., x_{2g}^*, y^*\right\} \in H^1(T_{f,1}).$$

Let $j : (T_{f,1}, \varnothing) \to (T_{f,1}, \partial T_{f,1})$ be the inclusion map. The long exact sequence of a pair shows $j_* : H_1(T_{f,1}) \to H_1(T_{f,1}, \partial T_{f,1})$ has kernel generated by y. So we have a basis for $H_1(T_{f,1}, \partial T_{f,1})$:

$$\{j_*(x_1), ..., j_*(x_{2g})\} \in H_1(T_{f,1}, \partial T_{f,1}).$$

And this gives a corresponding basis for $H_2(T_{f,1}) \cong H^1(T_{f,1}, \partial T_{f,1})$:

$$\{X_1, ..., X_{2g}\} \in H_2(T_{f,1}).$$

Let $\varepsilon : \mathbb{Z}[F] \to \mathbb{Z}$ be the augmentation map and let

$$\frac{\partial}{\partial \alpha_i} : \mathbb{Z}[F] \to \mathbb{Z}[F],\, 1 \leq i \leq 2g$$

be the Fox's free derivatives. Here $\mathbb{Z}[F]$ is the integral group ring of the free group F.

Finally, let $\mathfrak{X}$ denote the ring of formal power series in the noncommutative variables $t_1, ..., t_{2g}$, and let $\mathfrak{X}_k$ denote the submodule of $\mathfrak{X}$ corresponding to the degree k part. One can show F_k/F_{k+1} is a submodule of $\mathfrak{X}_k$, where the inclusion map is induced by

$$F_k \ni \zeta \longmapsto \sum_{j_1,\dots,j_k} \varepsilon \frac{\partial}{\partial \alpha_{j_1}} \cdots \frac{\partial}{\partial \alpha_{j_k}} (\zeta) t_{j_1} ... t_{j_k} \in \mathfrak{X}_k.$$

Then we have the following theorem.

Theorem 2.1 (Kitano) *There is a homomorphism $\tau_k : \mathcal{J}(k) \to \mathrm{Hom}(H_1, \mathfrak{X}_k)$ defined by letting $\tau_k(f)$ be the homomorphism*

$$x_i \longmapsto \sum_{j_1,\dots,j_k} \left\langle \left\langle x_{j_1}^*, ..., x_{j_k}^* \right\rangle, X_i \right\rangle t_{j_1} ... t_{j_k}$$

where $\langle \ , \ \rangle$ is the dual pairing of $H^2(T_{f,1})$ and $H_2(T_{f,1})$. Moreover, this homomorphism is the same as the Johnson homomorphism.

The canonical restriction $H^*(T_{f,1}, \partial T_{f,1}) \to H^*(T_{f,1})$ leads to the following theorem that gives a relation between the algebraic structure of the mapping class group $\Gamma_{g,1}$ and the topological structure of the mapping torus $T_{f,1}$.

Theorem 2.2 (Kitano) *For any $f \in \Gamma_{g,1}$, $f \in \mathcal{J}(k+1)$ if and only if all Massey products of length l of*

$$H^1(T_{f,1}, \partial T_{f,1}) \otimes \cdots \otimes H^1(T_{f,1}, \partial T_{f,1}) \to H^2(T_{f,1}, \partial T_{f,1}) \to H^2(T_{f,1})$$

vanish for any l with $1 < l \leq k$.

2.4 Morita's Refinement of τ_k

In this section we point out the work of Morita in [M1], where Johnson's homomorphism τ_k was refined so as to narrow the range of τ_k to a submodule $D_k(H_1)$ of $H_1 \otimes F_k/F_{k+1}$. This enhancement is obtained via a homomorphism

$$\tilde{\tau}_k : \mathcal{J}(k) \to H_3\left(\frac{F}{F_k}\right)$$

defined below. Recall that the homology of a group G is $H_i(G) \equiv H_i(K(G,1),\mathbb{Z})$, where $K(G,1)$ is an Eilenberg-MacLane space. (We determine the kernel of Morita's refinement in Corollary 5.19 below.)

Let $\zeta \in \pi_1(\Sigma_{g,1}) = F$ represent the homotopy class of a simple closed curve on $\Sigma_{g,1}$ parallel to the boundary $\partial\Sigma_{g,1}$. Now we choose a 2-chain $\sigma \in C_2(F)$ such that $\partial\sigma = -\zeta$. Since any $f \in \Gamma_{g,1}$ is required by definition to fix the boundary, we have $\partial(\sigma - f_\#(\sigma)) = -\zeta - (-\zeta) = 0$. Thus $\sigma - f_\#(\sigma)$ is a 2-cycle. Because $H_2(F)$ is trivial, there is a 3-chain $c_f \in C_3(F)$ such that $\partial c_f = \sigma - f_\#(\sigma)$. Note that, essentially, this is just a mapping cylinder construction. Let $\bar{c}_f$ denote the image of c_f in $C_3(F/F_k)$. If $f \in \mathcal{J}(k)$ then $f_\#$ acts as the identity on F/F_k. Thus we have $\partial\bar{c}_f = \overline{\sigma - f_\#(\sigma)} = \bar{\sigma} - f_\#(\bar{\sigma}) = 0$, and $\bar{c}_f$ is a 3-cycle. Finally define $[\bar{c}_f] \in H_3(F/F_k)$ to be the corresponding homology class, and we define Morita's homomorphism $\tilde{\tau}_k : \mathcal{J}(k) \to H_3(F/F_k)$ to be $\tilde{\tau}_k(f) = [\bar{c}_f]$. It is shown in [M1] that the homology class $[\bar{c}_f]$ does not depend on the choices that were made, and we refer you there for the details.

Now consider the extension

$$0 \to \frac{F_k}{F_{k+1}} \to \frac{F}{F_{k+1}} \to \frac{F}{F_k} \to 1,$$

and let $\left\{E^r_{p,q}\right\}$ be the Hochschild-Serre spectral sequence for the homology of this sequence. In particular, we have

$$E^2_{p,q} = H_p\left(\frac{F}{F_k}; H_q\left(\frac{F_k}{F_{k+1}}\right)\right).$$

Then we have the differential

$$d^2 : E^2_{3,0} = H_3\left(\frac{F}{F_k}\right) \to E^2_{1,1} = H_1\left(\frac{F}{F_k}; H_1\left(\frac{F_k}{F_{k+1}}\right)\right) \cong H_1 \otimes \frac{F_k}{F_{k+1}}.$$

Finally, the refinement of Johnson's homomorphism is given by the following theorem.

Theorem 2.3 (Morita) *The composition $d^2 \circ \tilde{\tau}_k$ coincides with Johnson's homomorphism τ_k so that the following diagram commutes.*

$$\begin{array}{ccc} & & H_3\left(\frac{F}{F_k}\right) \\ & \overset{\tilde{\tau}_k}{\nearrow} & \downarrow d^2 \\ \mathcal{J}(k) & \xrightarrow[\tau_k]{} & H_1 \otimes \frac{F_k}{F_{k+1}} \end{array}$$

Theorem 2.4 (Morita) *Let $D_k(H_1)$ be the submodule of $H_1 \otimes F_k/F_{k+1}$ defined to be the kernel of the natural surjection*

$$H_1 \otimes \frac{F_k}{F_{k+1}} \to \frac{F_{k+1}}{F_{k+2}}$$

given by the Lie bracket map $(w,\xi) \mapsto [w,\xi]$. *Then the image of the Johnson homomorphism* $\tau_k : \mathcal{J}(k) \to H_1 \otimes F_k/F_{k+1}$ *is contained in* $D_k(H_1)$ *so that we can write*

$$\tau_k : \mathcal{J}(k) \to D_k(H_1).$$

A short remark about this theorem is perhaps in order. It is known that the image of τ_2 is exactly equal to $D_2(H_1)$, and the image of τ_3 is a submodule of $D_3(H_1)$ of index a power of 2. Thus $\operatorname{Im}\tau_3$ and $D_3(H_1)$ have the same rank. However, for $k \geq 4$, k even, the rank of $\operatorname{Im}\tau_k$ is smaller than the rank of $D_k(H_1)$. Please see [M1] for more details.

Chapter 3

Birman-Craggs Homomorphism

As mentioned at the end of Section 2.1 the Johnson homomorphism τ_2 only detects the free abelian part of the abelianization of the Torelli group $\mathcal{J}(2)$, and some 2-torsion remains undetected. In this chapter we will say a word about this 2-torsion. In [BC] Birman and Craggs defined a (finite) collection of abelian quotients of $\mathcal{J}(2)$ given by homomorphisms onto $\mathbb{Z}_2$. Here we will give a description of these homomorphisms that is due to Johnson [J1]. This somewhat more tractable description is different than (yet equivalent to) Birman and Craggs' original definition, and it enabled Johnson to give the number of distinct Birman-Craggs homomorphisms.

Consider the surface $\Sigma_{g,1}$, and let $f \in \mathcal{J}(2)$. The definition of $\Gamma_{g,1}$ requires that f be the identity on $\partial\Sigma_{g,1}$. Thus f can easily be extended to a homeomorphism of the closed surface Σ_g. Let $h : \Sigma_g \to S^3$ be a Heegaard embedding of Σ_g into the 3-sphere S^3, i.e. Σ_g bounds handlebodies on both sides in S^3. Now cut S^3 open along $h(\Sigma_g)$ and reglue the two pieces using $f \in \mathcal{J}(2)$. The resulting manifold $S^3_{h,f}$ is a homology S^3, and its Rochlin invariant $\mu(S^3_{h,f}) \in \mathbb{Z}_2$ is defined.

In general, any closed, connected 3-manifold M, together with a fixed trivialization of its tangent bundle over the 2-skeleton, is the boundary of a 4-manifold W whose tangent bundle can be trivialized in a compatible fashion. If s denotes the choice of stable trivialization of the tangent bundle of M over the 2-skeleton, then the *Rochlin invariant* $\mu(M,s) \in \mathbb{Z}_{16}$ is defined to be the signature $\sigma(W)$ reduced modulo 16. If M happens to be a homology S^3 then s is unique and $\sigma(W)$ is divisible by 8. Thus $\mu(S^3_{h,f}) = \mu(S^3_{h,f}, s) = \sigma(W)$ can be considered an element of $\mathbb{Z}_2$. For a fixed Heegaard embedding $h : \Sigma_g \to S^3$, the *Birman-Craggs homomorphism* $\rho_h : \mathcal{J}(2) \to \mathbb{Z}_2$ is defined by $\rho_h(f) = \mu(S^3_{h,f})$.

By relating the Birman-Craggs homomorphisms to a $\mathbb{Z}_2$-quadratic form, Johnson was able to show the dependence of ρ_h on the embedding $h : \Sigma_g \to S^3$. We define the $\mathbb{Z}_2$-quadratic form $q : H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2$ as follows. Let $\langle\ ,\ \rangle$ be the Seifert linking form on $H_1(\Sigma_g;\mathbb{Z}_2)$ induced by $h : \Sigma_g \to S^3$ defined by letting $\langle x, y\rangle$ be the linking number (modulo 2) of $h(x)$ and $h(y)^+$ in S^3, where $h(y)^+$ is the positive push-off of $h(y)$ in the normal direction determined by the orientations of $h(\Sigma_g)$ and S^3. Define $q(x) = \langle x, x\rangle$, then it is a $\mathbb{Z}_2$-quadratic form on $H_1(\Sigma_g;\mathbb{Z}_2)$ induced by the embedding h. Because it is a quadratic form, q satisfies $q(x+y) = q(x) + q(y) + x\cdot y$, where $x \cdot y$ is the intersection pairing of $H_1(\Sigma_g;\mathbb{Z}_2)$. Let $\{x_i, y_i\}$, $1 \le i \le g$, denote the standard basis for $H_1(\Sigma_g;\mathbb{Z}_2)$, and the *Arf invariant* of Σ_g with respect to q is defined to be

$$\mathrm{Arf}(\Sigma_g, q) = \sum_{i=1}^{g} q(x_i)q(y_i) \pmod 2.$$

Johnson's main results from [J1] are as follows. Suppose $h_1,\ h_2 : \Sigma_g \to S^3$ are both Heegaard embeddings of the surface Σ_g.

Theorem 3.1 (Johnson) *The embeddings h_1 and h_2 induce the same mod 2 self-linking form if and only if the Birman-Craggs homomorphisms ρ_{h_1} and ρ_{h_2} are equal.*

Therefore the homomorphism $\rho_h : \mathcal{J}(2) \to \mathbb{Z}_2$ only depends on the quadratic form q induced by h, and we replace the notation ρ_h with ρ_q to emphasize this fact. Moreover, the $\mathbb{Z}_2$-quadratic forms q which are induced by a Heegaard embedding h are exactly those that satisfy $\mathrm{Arf}(\Sigma_g, q) = 0$. Thus we are able to enumerate $\{\rho_q\}$.

Corollary 3.2 (Johnson) *There are precisely* $2^{g-1}(2^g + 1)$ *distinct Birman-Craggs homomorphisms* $\rho_q : \mathcal{J}(2) \to \mathbb{Z}_2$.

Johnson also provided a means of computing ρ_q in terms of the Arf invariant.

(1) If $f \in \mathcal{J}(2)$ is a Dehn twist about a bounding simple closed curve C, then

$$\rho_q(f) = \mathrm{Arf}(\Sigma', q|_{\Sigma'}),$$

where Σ' is a subsurface of Σ_g bounded by C.

(2) If $f \in \mathcal{J}(2)$ is a composition of Dehn twists about cobounding curves C_1 and C_2, then

$$\rho_q(f) = \begin{cases} 0 & \text{if } q(C_1) = q(C_2) = 1 \\ \mathrm{Arf}(\Sigma', q|_{\Sigma'}) & \text{if } q(C_1) = q(C_2) = 0 \end{cases}$$

where Σ' is a subsurface of Σ_g cobounded by C_1 and C_2.

For genus $g = 2$ surfaces, the Torelli group $\mathcal{J}(2)$ is generated by the collection of all Dehn twists about bounding simple closed curves. For genus $g \geq 3$, $\mathcal{J}(2)$ is generated by the collection of all Dehn twists about genus 1 cobounding pairs of simple closed curves, i.e. pairs of non-bounding, disjoint, homologous simple closed curves that together bound a genus 1 subsurface. Thus the list above is sufficient for computing $\rho_q(f)$ for any $f \in \mathcal{J}(2)$.

Chapter 4

Abelianization of the Torelli Group

We are now prepared to say something about the abelianization

$$H_1(\mathcal{J}(2);\mathbb{Z}) \cong \frac{\mathcal{J}(2)}{[\mathcal{J}(2),\mathcal{J}(2)]}$$

of the Torelli group $\mathcal{J}(2)$. In fact, the main result of Johnson in [J6] is that the Johnson homomorphism

$$\tau_2 : \mathcal{J}(2) \to D_2(H_1)$$

and the totality of the Birman-Craggs homomorphisms

$$\rho_q : \mathcal{J}(2) \to \mathbb{Z}_2$$

completely determine $H_1(\mathcal{J}(2);\mathbb{Z})$

On the one hand, we have the composition

$$\frac{\mathcal{J}(2)}{[\mathcal{J}(2),\mathcal{J}(2)]} \twoheadrightarrow \frac{\mathcal{J}(2)}{\mathcal{J}(3)} \xrightarrow{\cong} D_2(H_1)$$

where the first map is the projection given by the fact that $[\mathcal{J}(2), \mathcal{J}(2)] \subset \mathcal{J}(3)$ and the second map is given by τ_2. After we tensor with the rationals $\mathbb{Q}$, Johnson shows that we obtain an isomorphism

$$\frac{\mathcal{J}(2)}{[\mathcal{J}(2), \mathcal{J}(2)]} \otimes \mathbb{Q} \overset{\cong}{\longrightarrow} \frac{\mathcal{J}(2)}{\mathcal{J}(3)} \otimes \mathbb{Q} \overset{\cong}{\longrightarrow} D_2(H_1) \otimes \mathbb{Q}.$$

Thus we have $H_1(\mathcal{J}(2); \mathbb{Q}) \cong \mathcal{J}(2)/\mathcal{J}(3) \otimes \mathbb{Q}$.

On the other hand, consider the totality of the Birman-Craggs homomorphisms $\{\rho_q\}$, and let

$$\mathcal{C} = \bigcap_q \ker \rho_q$$

be the common kernel of all ρ_q for all q which satisfy $\mathrm{Arf}(\Sigma_g, q) = 0$. Also let $\mathcal{J}(2)^2$ represent the subgroup generated by all squares in $\mathcal{J}(2)$, and let $\mathcal{O}_q$ be the subgroup of the mapping class group $\Gamma_{g,1}$ which acts trivially on $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$. That is, $\mathcal{O}_q$ consists of those homeomorphisms which preserve the quadratic form q. Then, by using the theory of Boolean quadratic and cubic forms, Johnson showed that

$$\mathcal{C} = \mathcal{J}(2)^2 = [\mathcal{O}_q, \mathcal{J}(2)]\,.$$

Finally he showed that the commutator subgroup of $\mathcal{J}(2)$ is given by

$$[\mathcal{J}(2), \mathcal{J}(2)] = \mathcal{C} \cap \ker \tau_2 = \mathcal{C} \cap \mathcal{J}(3).$$

Thus we can completely determine $H_1(\mathcal{J}(2); \mathbb{Z}) \cong \mathcal{J}(2)/\left[\mathcal{J}(2), \mathcal{J}(2)\right]$ from the homomorphisms $\{\tau_2, \rho_q\}$.

Chapter 5

A Bordism Representation of the Mapping Class Group

5.1 The Bordism Group $\Omega_3(X, A)$

Let (X, A) be a pair of topological spaces $A \subseteq X$. The *3-dimension oriented relative bordism group* $\Omega_3(X, A)$ is defined to be the set of *bordism classes* of triples $(M, \partial M, \phi)$ consisting of a compact, oriented 3-manifold M with boundary ∂M and a continuous map $\phi : (M, \partial M) \to (X, A)$. The triples $(M_0, \partial M_0, \phi_0)$ and $(M_1, \partial M_1, \phi_1)$ are equivalent, or *bordant over* (X, A), if there exists a triple $(W, \partial W, \Phi)$ consisting of a compact, oriented 4-manifold W with boundary $\partial W = (M_0 \amalg -M_1) \cup_{\partial M} M$ and a continuous map $\Phi : (W, \partial W) \to (X, A)$ satisfying $\Phi|_{M_i} = \phi_i$ and $\Phi(M) \subset A$. We also require that $\partial M = \partial M_0 \amalg -\partial M_1$ so that ∂W is a closed 3-manifold.

A triple $(M, \partial M, \phi)$ is said to be *null-bordant* (or trivial) over (X, A) if it bounds $(W, \partial W, \Phi)$, that is, if it is bordant to the empty set $\varnothing$. The set $\Omega_3(X, A)$ forms a group with the operation of disjoint union and identity element $\varnothing$. In the case

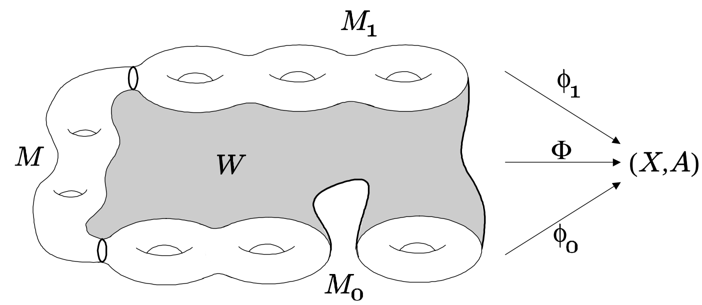


Figure 5.1: A relative bordism over (X, A)

that $A = \varnothing$, we may write $\Omega_3(X) = \Omega_3(X, \varnothing)$ and restrict our definition to pairs $(M, \phi) = (M, \varnothing, \phi)$ of closed, oriented 3-manifolds.

5.2 A Bordism Invariant of $\mathcal{J}(k)$

The purpose of this section is to analyze $\mathcal{J}(k)$ from the point of view of bordism theory. Let $F = \pi_1(\Sigma_{g,1})$ as before, and consider the pair $(K(F/F_k, 1), \zeta)$, where $K(F/F_k, 1)$ is an Eilenberg-MacLane space and $\zeta \subset K(F/F_k, 1)$ is an S^1 corresponding to the image of $\partial\Sigma_{g,1}$ under a continuous map $\Sigma_{g,1} \to K(F/F_k, 1)$ induced by the canonical projection $F \twoheadrightarrow F/F_k$. We denote the bordism group over $(K(F/F_k, 1), \zeta)$ by $\Omega_3(F/F_k, \zeta)$. Moreover, we have an isomorphism $j_* : \Omega_3(F/F_k) \to \Omega_3(F/F_k, \zeta)$ induced by the inclusion map $j : (K(F/F_k, 1), \varnothing) \to (K(F/F_k, 1), \zeta)$. We will make use of both of these groups in what follows, but our main focus will be on the group $\Omega_3(F/F_k)$.

Below, in Theorem 5.2, we define a homomorphism

$$\sigma_k : \mathcal{J}(k) \to \Omega_3\left(\frac{F}{F_k}\right)$$

whose kernel is $\ker\sigma_k = \mathcal{J}(2k-1)$. Thus the image is

$$\text{image}(\sigma_k) \cong \frac{\mathcal{J}(k)}{\mathcal{J}(2k-1)}.$$

We have already seen that the image of the Johnson homomorphism τ_k satisfies

$$\text{image}(\tau_k) \cong \frac{\mathcal{J}(k)}{\mathcal{J}(k+1)}.$$

However, since we know that $[\mathcal{J}(k), \mathcal{J}(k)] \subset \mathcal{J}(2k-1) \subset \mathcal{J}(k+1)$, the image of this new homomorphism σ_k is, in general, much closer to the abelianization of $\mathcal{J}(k)$.

Consider a surface homeomorphism $f \in \mathcal{J}(k)$ for some $k \geq 2$. As before let $T_{f,1}$ be the mapping torus of f, i.e. $\Sigma_{g,1} \times [0,1]$ with $x \times \{0\}$ glued to $f(x) \times \{1\}$. The boundary $\partial T_{f,1}$ of $T_{f,1}$ is the torus $\partial\Sigma_{g,1} \times S^1$, and the mapping torus $T_{f,1}$ is an (oriented) homology $\Sigma_{g,1} \times S^1$. Fixing a basis $\{\alpha_1, ..., \alpha_{2g}\}$ for the free group $F = \pi_1(\Sigma_{g,1})$ gives a presentation of $\pi_1(T_{f,1})$:

$$\pi_1(T_{f,1}) = \left\langle \alpha_1, ..., \alpha_{2g}, \gamma \mid [\alpha_1, \gamma]\, f_*(\alpha_1)\alpha_1^{-1}, ..., [\alpha_{2g}, \gamma]\, f_*(\alpha_{2g})\alpha_{2g}^{-1} \right\rangle$$

where γ represents a generator of $\pi_1(S^1)$. We now wish to obtain a closed 3-manifold from $T_{f,1}$ by filling in its boundary. Let $T_f^\gamma = T_{f,1}^\gamma$ be the result of performing a Dehn filling along a curve on $\partial T_{f,1}$ represented by the homotopy class γ. That is, T_f^γ is obtained by filling in the torus $\partial T_{f,1} \simeq \partial\Sigma_{g,1} \times S^1$ with the solid torus $\partial\Sigma_{g,1} \times D^2$. Then we also have a presentation for $\pi_1(T_f^\gamma)$:

$$\pi_1(T_f^\gamma) = \left\langle \alpha_1, ..., \alpha_{2g} \mid f_*(\alpha_1)\alpha_1^{-1}, ..., f_*(\alpha_{2g})\alpha_{2g}^{-1} \right\rangle$$

Note that if f is isotopic to the identity, then T_f^γ is homeomorphic to the connected sum $\overset{2g}{\#}(S^1 \times S^2)$.

Now for all $m \leq k$ we can define $\phi_{f,m} : (T_{f,1}, \partial T_{f,1}) \to (K(F/F_m, 1), \zeta)$ to be a continuous map induced by the canonical epimorphism

$$\pi_1(T_{f,1}) \twoheadrightarrow \frac{\pi_1(T_{f,1})}{\left\langle \gamma, (\pi_1(T_{f,1}))_m \right\rangle} \cong \frac{F}{F_m}$$

where the isomorphism requires the fact that $f \in \mathcal{J}(k) \subset \mathcal{J}(m)$ (see Lemma 5.1 below.) Also, since we kill the homotopy class γ in our construction of T_f^γ, the map $\phi_{f,m}$ extends to a continuous map $\phi_{f,m}^\gamma : T_f^\gamma \to K(F/F_m, 1)$, and $\phi_{f,m}^\gamma$ induces the canonical epimorphism

$$\pi_1(T_f^\gamma) \twoheadrightarrow \frac{\pi_1(T_f^\gamma)}{\left(\pi_1(T_f^\gamma)\right)_m} \cong \frac{F}{F_m}.$$

Moreover, we have the following lemma.

Lemma 5.1 *The following are equivalent:*

(a) $f \in \mathcal{J}(m)$,

(b) $\frac{\pi_1(T_f^\gamma)}{\left(\pi_1(T_f^\gamma)\right)_m} \cong \frac{F}{F_m}$ *and* $\frac{\pi_1(T_{f,1})}{\left\langle \gamma, \left(\pi_1(T_{f,1})\right)_m \right\rangle} \cong \frac{F}{F_m}$, *and*

(c) the continuous maps $\phi_{f,m}^\gamma$ *and* $\phi_{f,m}$ *exist as defined.*

Proof. This is an obvious fact, but we wish to emphasize it because of the important role it will play later.

(a) $\Longleftrightarrow$ *(b).* If $f \in \mathcal{J}(m)$ then the relations $[\alpha_i, \gamma]\, f_*(\alpha_i)\alpha_i^{-1}$ in $\pi_1(T_{f,1})$ become trivial modulo $\left\langle \gamma, (\pi_1(T_{f,1}))_m \right\rangle$ since f_* acts as the identity on F/F_m, and we clearly have

a homomorphism (in fact, an isomorphism.) On the other hand, no such homomorphism exists if $f \notin \mathcal{J}(m)$ because the relations $[\alpha_i, \gamma]\, f_*(\alpha_i)\alpha_i^{-1} \equiv f_*(\alpha_i)\alpha_i^{-1} \bmod(\gamma)$ are certainly not trivial modulo $(\pi_1(T_{f,1}))_m$.

(b) $\Longleftrightarrow$ *(c).* It is a well-known property of Eilenberg-MacLane spaces that continuous maps into them are in one-to-one correspondence with homomorphisms into their fundamental group. (See [Wh] Theorem V.4.3.) Thus $\phi^{\gamma}_{f,m}$ (and similarly for $\phi_{f,m}$) is defined if and only if the homomorphism

$$\pi_1(T_{f,1}) \twoheadrightarrow \frac{\pi_1(T_{f,1})}{\langle \gamma, (\pi_1(T_{f,1}))_m \rangle} \cong \frac{F}{F_m}$$

exists. □

Let us now consider the pair $(T^{\gamma}_f, \phi^{\gamma}_{f,k}) \in \Omega_3(F/F_k)$ (and analogously the triple $(T_{f,1}, \partial T_{f,1}, \phi_{f,k}) \in \Omega_3(F/F_k, \zeta)$.) We introduce a new homomorphism giving a representation of $\mathcal{J}(k)$ which is very geometric in nature.

Theorem 5.2 *The map*

$$\sigma_k : \mathcal{J}(k) \to \Omega_3\left(\frac{F}{F_k}\right)$$

defined by $\sigma_k(f) = (T^{\gamma}_f, \phi^{\gamma}_{f,k})$ *is a well-defined homomorphism.*

We also point out that one can similarly define a homomorphism into the relative bordism group $\mathcal{J}(k) \to \Omega_3(F/F_k, \zeta)$ which sends a mapping class $f \in \mathcal{J}(k)$ to $(T_{f,1}, \partial T_{f,1}, \phi_{f,k})$. However, we will mainly focus on the homomorphism given in Theorem 5.2.

Proof. Consider two homeomorphisms $f, g \in \mathcal{J}(k)$ for the oriented surface Σ_1 with one boundary component. If f and g are isotopic, i.e. they represent the same mapping class, then of course T_f^γ and T_g^γ are homeomorphic and $\left(T_f^\gamma, \phi_{f,k}^\gamma\right)$ and $\left(T_g^\gamma, \phi_{g,k}^\gamma\right)$ are bordant. Thus σ_k is certainly well-defined.

To show σ_k is indeed a homomorphism we need to show that $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \amalg \left(T_g^\gamma, \phi_{g,k}^\gamma\right)$ is bordant to $\left(T_{f\circ g}^\gamma, \phi_{f\circ g,k}^\gamma\right)$ in $\Omega_3(F/F_k)$ for any mapping classes $f, g \in \mathcal{J}(k)$. To do so, we simply construct a bordism, i.e. we build a 4-manifold W and continuous map $\Phi : W \to K(F/F_k, 1)$ with boundary given by

$$(\partial W, \Phi|_{\partial W}) = \left[\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \amalg \left(T_g^\gamma, \phi_{g,k}^\gamma\right)\right] \amalg - \left(T_{f\circ g}^\gamma, \phi_{f\circ g,k}^\gamma\right).$$

We begin by first constructing a 4-manifold between the mapping tori $T_{f,1} \amalg T_{g,1}$ and $T_{f\circ g,1}$. Recall that

$$T_{f,1} = \frac{\Sigma_1 \times [0,1]}{(x,0) \sim (f(x),1)}.$$

Moreover, we can also consider $T_{f\circ g,1}$ in "blocks" as follows and as depicted in Figure 5.2.

$$T_{f\circ g,1} = \frac{\Sigma_1 \times [0,1]}{(x,0) \sim (f(g(x)),1)} \cong \frac{\left(\Sigma_1 \times \left[0, \frac{1}{2}\right]\right) \cup \left(\Sigma_1 \times \left[\frac{1}{2}, 1\right]\right)}{(x,0) \sim (f(x),1), \left(x, \frac{1}{2}\right) \sim \left(g(x), \frac{1}{2}\right)}$$

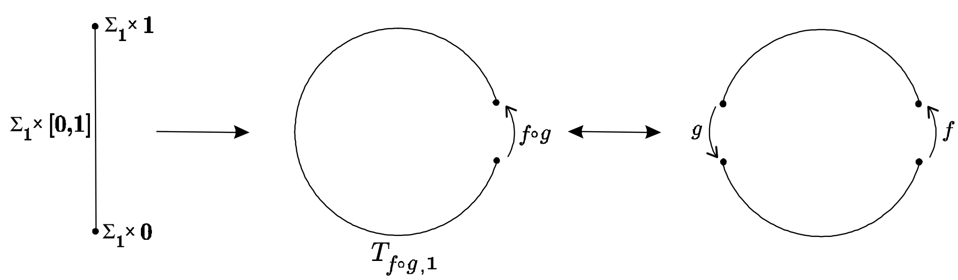


Figure 5.2: $T_{f\circ g,1}$ considered in "blocks"

We can assume there is a product neighborhood of $\Sigma_1 \times \left\{\frac{1}{2}\right\}$ in $T_{f,1}$, i.e. a cylinder $\left(\Sigma_1 \times \left\{\frac{1}{2}\right\}\right) \times [-\varepsilon, \varepsilon]$. Let $V = (T_{f,1} \amalg T_{g,1}) \times [0,1]$. Then V has boundary

$$\partial V = (T_{f,1} \amalg T_{g,1}) \times \{0\} \cup - (T_{f,1} \amalg T_{g,1}) \times \{1\} \cup (\partial T_{f,1} \amalg \partial T_{g,1}) \times [0,1] .$$

Now consider the piece $(T_{f,1} \amalg T_{g,1}) \times \{1\}$ of ∂V and attach a 4-dimensional "strip" $\Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]$ to $(T_{f,1} \amalg T_{g,1}) \times \{1\}$ by gluing $\Sigma_1 \times [-\varepsilon, \varepsilon] \times \{-\delta\}$ to the neighborhood $\left(\Sigma_1 \times \left\{\frac{1}{2}\right\}\right) \times [-\varepsilon, \varepsilon]$ in $T_{f,1}$ and gluing $\Sigma_1 \times [-\varepsilon, \varepsilon] \times \{\delta\}$ to the neighborhood $\left(\Sigma_1 \times \left\{\frac{1}{2}\right\}\right) \times [-\varepsilon, \varepsilon]$ in $T_{g,1}$. Let V' be the result of this gluing, then

$$\begin{aligned} \partial V' \;=\; & ((T_{f,1} \amalg T_{g,1}) \times \{0\}) \cup (- (T_{f\circ g,1}) \times \{1\}) \cup ((\partial T_{f,1} \amalg \partial T_{g,1}) \times [0,1]) \\ & \cup (\partial \Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]) . \end{aligned}$$

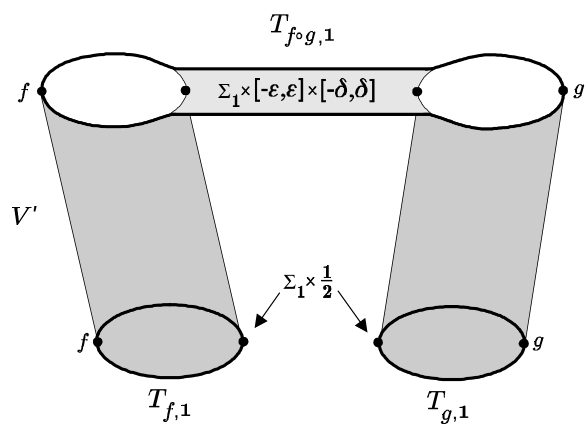


Figure 5.3: The 4-manifold V'

We now fill in the boundary component $(\partial T_{f,1} \amalg \partial T_{g,1}) \times [0,1]$ with

$$\left(\left(\partial \Sigma_1 \times D^2\right) \amalg \left(\partial \Sigma_1 \times D^2\right)\right) \times [0,1] \qquad (\star)$$

to obtain a new 4-manifold W. At one end, this has the effect of filling in the boundary of $(T_{f,1} \amalg T_{g,1}) \times \{0\}$, thus creating $\left(T_f^\gamma \amalg T_g^\gamma\right) \times \{0\}$. At the other end, we had already filled in *some* of the boundary of $T_{f\circ g}^\gamma \times \{1\}$ with $(\partial\Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta])$ above, and the filling by $(\star)$ has the effect of filling in the rest of the boundary of $T_{f\circ g,1} \times \{1\}$. Thus we have actually created $T_{f\circ g}^\gamma \times \{1\}$. Therefore we have created a 4-manifold W with boundary

$$\partial W = \left(T_f^\gamma \amalg T_g^\gamma\right) \amalg -T_{f\circ g}^\gamma.$$

Also, the continuous map $\phi_{f,k} \amalg \phi_{g,k}$ clearly extends over $V = (T_{f,1} \amalg T_{g,1}) \times [0,1]$. It is also easy to see that it extends over V' as well since $\Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]$ deformation retracts to Σ_1. Finally it extends to a continuous map $\Phi : W \to K(F/F_k, 1)$ in a similar way that $\phi_{f,k}$ extends to $\phi_{f,k}^\gamma$. Therefore $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \amalg \left(T_g^\gamma, \phi_{g,k}^\gamma\right)$ is bordant to $\left(T_{f\circ g}^\gamma, \phi_{f\circ g,k}^\gamma\right)$ in $\Omega_3(F/F_k)$, and we have completed the proof of Theorem 5.2. □

Notice that if a surface homeomorphism f is isotopic to the identity then its mapping class is in $\mathcal{J}(k)$ for all k, and $\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right) = \left(T_{id,1}, \partial T_{id,1}, \phi_{id,k}\right)$ and $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) = \left(T_{id}^\gamma, \phi_{id,k}^\gamma\right)$ are null-bordant in $\Omega_3(F/F_k, \zeta)$ and $\Omega_3(F/F_k)$, respectively, since they each bound $\Sigma_{g,1} \times D^2$ and the respective maps clearly extend. (The definition of relative bordism requires an "extra" boundary piece so that the boundary of the 4-manifold is a closed 3-manifold. In the case of $(T_{id,1}, \partial T_{id,1})$, the extra piece is simply the solid torus $\partial\Sigma_{g,1} \times D^2$ used to construct T_{id}^γ.) It seems logical to ask when $\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right) \in \Omega_3(F/F_k, \zeta)$ and $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \in \Omega_3(F/F_k)$ are null-bordant for more general $f \in \mathcal{J}(k)$. That is, what is the kernel of σ_k? This is answered by the following theorem.

Theorem 5.3 *$\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right) \in \Omega_3(F/F_k, \zeta)$ and $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \in \Omega_3(F/F_k)$ are trivial if and only if $f \in \mathcal{J}(2k-1)$.*

Corollary 5.4 *The kernel of the homomorphism* σ_k *is* $\mathcal{J}(2k-1)$. $\square$

We also have the following generalization of Theorem 5.3 which is a corollary to the proof of Theorem 5.2.

Corollary 5.5 *Consider* $f, g \in \mathcal{J}(k)$. *Then the following are equivalent:*

(a) $f \circ g^{-1} \in \mathcal{J}(2k-1)$,

(b) $\left(T_f^\gamma, \phi_{f,k}^\gamma\right)$ *is bordant to* $\left(T_g^\gamma, \phi_{g,k}^\gamma\right)$ *in* $\Omega_3(F/F_k)$,

(c) $\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right)$ *is bordant to* $\left(T_{g,1}, \partial T_{g,1}, \phi_{g,k}\right)$ *in* $\Omega_3(F/F_k, \zeta)$.

Proof. Suppose we have $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) = \left(T_g^\gamma, \phi_{g,k}^\gamma\right)$ in $\Omega_3(F/F_k)$. This is equivalent to having $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \amalg \left(T_{g^{-1}}^\gamma, \phi_{g^{-1},k}^\gamma\right) = \left(T_g^\gamma, \phi_{g,k}^\gamma\right) \amalg \left(T_{g^{-1}}^\gamma, \phi_{g^{-1},k}^\gamma\right)$. However, we showed in the proof of Theorem 5.2 that this is equivalent to $\left(T_{f\circ g^{-1}}^\gamma, \phi_{f\circ g^{-1},k}^\gamma\right) = \left(T_{g\circ g^{-1}}^\gamma, \phi_{g\circ g^{-1},k}^\gamma\right)$. The latter is just $\left(T_{id}^\gamma, \phi_{id,k}^\gamma\right)$, which is nullbordant. Thus Theorem 5.3 says that this is equivalent to $f \circ g^{-1} \in \mathcal{J}(2k-1)$. The equivalence of (c) is proved similarly. $\square$

Proof of Theorem 5.3. We prove the theorem for the pair $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) \in \Omega_3(F/F_k)$, and the proof for the triple $\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right) \in \Omega_3(F/F_k, \zeta)$ is completely analogous. Suppose $f \in \mathcal{J}(m)$, then for $l \leq m$ let $\pi_{m,l} : K(F/F_m, 1) \to K(F/F_l, 1)$ be the projection map such that $\phi_{f,l}^\gamma = \pi_{m,l} \circ \phi_{f,m}^\gamma$.

($\Longleftarrow$). Let us first suppose that $f \in \mathcal{J}(2k-1)$. Then the pair $\left(T_f^\gamma, \phi_{f,2k-1}^\gamma\right)$ is defined and is an element of $\Omega_3(F/F_{2k-1})$. The following lemma is due to K. Igusa and K. Orr ([IO], Theorem 6.7.)

Lemma 5.6 (Igusa-Orr) *Let* $\left(\pi_{m,k}\right)_*$ *be the induced map on* H_3. *Let* $x \in H_3(F/F_m)$. *Then* $x \in \ker\left(\pi_{m,k}\right)_*$ *if and only if* $x \in \operatorname{Image}\left(\pi_{2k-1,m}\right)_*$ *for* $k \leq m \leq 2k-1$. *In* *particular, the homomorphism*

$$(\pi_{2k-1,k})_* : H_3\left(\frac{F}{F_{2k-1}}\right) \to H_3\left(\frac{F}{F_k}\right)$$

is trivial.

We have the following corollary.

Corollary 5.7 *The homomorphism*

$$(\pi_{2k-1,k})_* : \Omega_3\left(\frac{F}{F_{2k-1}}\right) \to \Omega_3\left(\frac{F}{F_k}\right)$$

is trivial. Moreover, a bordism class is in $\ker(\pi_{m,k})_*$ *if and only if it lies in the image of* $(\pi_{2k-1,m})_*$ *for* $k \leq m \leq 2k-1$.

Proof. In general, $\Omega_n(X, A)$ is the n-dimensional bordism group, and it is an extraordinary homology theory. Using the Atiyah-Hirzebruch spectral sequence, (see G. Whitehead [Wh] for details,) one can express $\Omega_n(X, A)$ in terms of ordinary homology with coefficient group Ω_q, where $\Omega_q = \Omega_q(\cdot)$ is the bordism group of a single point. In particular, $E^2_{p,q} \cong H_p(X, A; \Omega_q)$ and the boundary operator is $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$, and $\Omega_n(X, A)$ is built using $H_p(X, A; \Omega_q)$ with $p+q = n$. Now $\Omega_0 \cong \mathbb{Z}$ and Ω_1, Ω_2, and Ω_3 are all trivial. So in the case $n = 3$ we have $\Omega_3(X, A) \cong H_3(X, A; \Omega_0) \cong H_3(X, A)$. In fact, the isomorphism is given by $(M, \partial M, \phi) \longmapsto \phi_*([M, \partial M])$ where $[M, \partial M]$ denotes the fundamental class in $H_3(M, \partial M)$. Of course it follows directly that $\Omega_3(F/F_k) \cong H_3(F/F_k)$ (and $\Omega_3(F/F_k, \zeta) \cong H_3(F/F_k, \zeta)$,) and we have the following commutative diagram:

$$\begin{array}{ccccc} H_3\left(\dfrac{F}{F_{2k-1}}\right) & \xrightarrow{(\pi_{2k-1,m})_*} & H_3\left(\dfrac{F}{F_m}\right) & \xrightarrow{(\pi_{m,k})_*} & H_3\left(\dfrac{F}{F_k}\right) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \Omega_3\left(\dfrac{F}{F_{2k-1}}\right) & \xrightarrow{(\pi_{2k-1,m})_*} & \Omega_3\left(\dfrac{F}{F_m}\right) & \xrightarrow{(\pi_{m,k})_*} & \Omega_3\left(\dfrac{F}{F_k}\right) \end{array}$$

Since the map $(\pi_{2k-1,k})_*$ on H_3 is the zero-homomorphism, the conclusion of the first part of the corollary is proved. The proof of the latter part is also immediate. □

The image of $\left(T_f^\gamma, \phi_{f,2k-1}^\gamma\right)$ under $(\pi_{2k-1,k})_* : \Omega_3(F/F_{2k-1}) \to \Omega_3(F/F_k)$ is

$$(\pi_{2k-1,k})_* \left(T_f^\gamma, \phi_{f,2k-1}^\gamma\right) = \left(T_f^\gamma, \pi_{2k-1,k} \circ \phi_{f,2k-1}^\gamma\right) = \left(T_f^\gamma, \phi_{f,k}^\gamma\right),$$

and Corollary 5.7 tells us that this image is trivial in $\Omega_3(F/F_k)$. Thus the condition $f \in \mathcal{J}(2k-1)$ is certainly sufficient.

$(\implies)$. The proof of the necessity of $f \in \mathcal{J}(2k-1)$ is much more subtle. If we assume that $\left(T_f^\gamma, \phi_{f,k}^\gamma\right)$ is trivial in $\Omega_3(F/F_k)$, then Corollary 5.7 tells us that there is a pair $(M, \phi) \in \Omega_3(F/F_{2k-1})$ that gets sent to $\left(T_f^\gamma, \phi_{f,k}^\gamma\right)$, but we do not know anything more than that. We want to show that $\phi_{f,2k-1}^\gamma$ is defined, and by Lemma 5.1 we may achieve the desired conclusion $f \in \mathcal{J}(2k-1)$.

Lemma 5.8 (Cochran-Gerges-Orr) *Let M be any oriented manifold such that $\pi_1(M) = G$, and suppose F is a free group. Then for $k > 1$, $G/G_k \cong F/F_k$ if and only if $H_1(M)$ is torsion-free and all Massey products for $H^1(M)$ of length less than k vanish. Under the latter conditions, any isomorphism $G/G_{k-1} \cong F/F_{k-1}$ extends to $G/G_k \cong F/F_k$.*

Proof. If $G/G_k \cong F/F_k$ then there is a continuous map $\phi : M \to K(F/F_k, 1)$ that induces an isomorphism $\phi^* : H^1(M) \to H^1(F/F_k)$ and $H_1(M)$ is clearly torsion-free. In [Or] (Lemma 16) it is shown that Massey products for $H^1(F/F_k)$ of length less than k vanish and length k Massey products generate $H^2(F/F_k)$. Consider $x_i \in H^1(F/F_k)$, then $\langle x_1, ..., x_n \rangle = 0$ for all $n < k$. Also, the naturality of Massey products (see (2.2.2)) tells us that $\phi^* \langle x_1, ..., x_n \rangle \subset \langle \phi^* x_1, ..., \phi^* x_n \rangle$. Thus for $n < k$ we certainly have $0 \in \langle \phi^* x_1, ..., \phi^* x_n \rangle$. However, the uniqueness of Massey products given in (2.2.1) tells us that the first nonzero Massey product is uniquely defined, and we conclude that $0 = \langle \phi^* x_1, ..., \phi^* x_n \rangle$ for $n < k$. Therefore all Massey products for $H^1(M)$ of length less than k are zero.

On the other hand, if $H_1(M)$ is torsion-free and all Massey products for $H^1(M)$ of length less than k vanish then we easily see that $H_1(M) \cong G/G_2 \cong F/F_2$. Now assume by induction that $G/G_{k-1} \cong F/F_{k-1}$, and let $\psi : F \to G$ be a homomorphism that induces this isomorphism. We will extend this isomorphism to $G/G_k \cong F/F_k$. It is sufficient to show that $G_{k-1}/G_k \cong F_{k-1}/F_k$. We have the following commutative diagram

$$\begin{array}{ccccccc} & 0 & \longrightarrow & H_2\left(\dfrac{F}{F_{k-1}}\right) & \xrightarrow{\cong} & \dfrac{F_{k-1}}{F_k} & \longrightarrow & 0 \\ & & & \cong\Big\downarrow \psi_* & & \Big\downarrow \psi_* & & \\ H_2(G) & \xrightarrow{\pi_*} & & H_2\left(\dfrac{G}{G_{k-1}}\right) & \longrightarrow & \dfrac{G_{k-1}}{G_k} & \longrightarrow & 0 \end{array}$$

in which the horizontal maps are exact sequences. The fact that the sequences are exact is a result of J. Stallings [St]. This diagram shows us that it is sufficient to show that $\pi_* : H_2(G) \to H_2(G/G_{k-1})$ is trivial. However, since $H_2(M) \twoheadrightarrow H_2(G)$ is onto, we need only show that $\pi_* : H_2(M) \to H_2(G/G_{k-1})$ is trivial. As mentioned above, length $k-1$ Massey products $\langle x_1, ..., x_{k-1}\rangle$ generate $H^2(G/G_{k-1}) \cong H^2(F/F_{k-1})$. Then $\pi^* \langle x_1, ..., x_{k-1}\rangle = \langle \pi^* x_1, ..., \pi^* x_{k-1}\rangle = 0$ since length $k-1$ Massey products vanish for M. Therefore π^* and π_* are trivial homomorphisms, and the conclusion follows. □

A slightly more general version of the following lemma is proved in [CGO] (Theorem 4.2), and we include a proof here for your convenience.

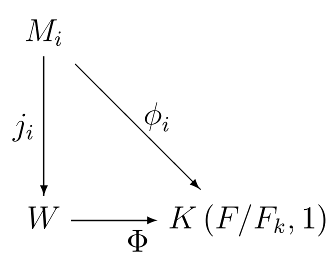
Lemma 5.9 (Cochran-Gerges-Orr) *Suppose M_0 and M_1 are closed, oriented 3-manifolds with $\pi_1(M_0) = G_0$ and $\pi_1(M_1) = G_1$. Also suppose there is an epimorphism $\psi : G_1 \to G_0/(G_0)_k$. Let $\phi_0 : M_0 \to K(G_0/(G_0)_k, 1)$ and $\phi_1 : M_1 \to K(G_0/(G_0)_k, 1)$ be continuous maps so that $(\phi_1)_* = \psi$ and $(M_0, \phi_0) = (M_1, \phi_1)$ in $\Omega_3(G_0/(G_0)_k)$. Then (M_0, ϕ_0) and (M_1, ϕ_1) are bordant over $K(G_0/(G_0)_k, 1)$ via a 4-manifold with only 2-handles (rel M_0) whose attaching circles lie in $(G_0)_k$.*

Proof. Since (M_0, ϕ_0) and (M_1, ϕ_1) are bordant in $\Omega_3(G_0/(G_0)_k)$, we know there exists a compact, oriented 4-manifold W and a continuous map $\Phi : W \to K(G_0/(G_0)_k, 1)$ such that $\partial(W, \Phi) = (M_0, \phi_0) \amalg (-M_1, \phi_1)$. Φ_* is already a surjection on π_1, and we can make it an injection by performing surgery on loops in W. Thus we may assume Φ_* is an isomorphism. Now we choose a handlebody structure for W relative to M_0 with no 0-handles or 4-handles. We then get rid of the 1-handles by trading them for 2-handles, i.e. we perform a surgery along a loop c passing over the 1-handles in the interior of W. In a similar manner, we can get rid of the 3-handles by thinking of them as 1-handles relative to M_1. Let V be the result of this handle swapping. We want to make sure Φ extends to V, so because Φ_* is an isomorphism it is necessary to make sure c was null-homotopic in W since it is null-homotopic in V. However, since

$(\phi_0)_*$ is surjective and c is in the interior of W, we can alter c by a loop in M_0 so that the altered c is null-homotopic in W. Thus we may assume that the 2-handles are attached along loops c in $(G_0)_k$. □

Lemma 5.10 *Let M_i and G_i $(i = 0, 1)$ be as in Lemma 5.9. For some free group F suppose that $\phi_0 : M_0 \to K(F/F_k, 1)$ and $\phi_1 : M_1 \to K(F/F_k, 1)$ are continuous maps such that ϕ_0 induces an isomorphism $G_0/(G_0)_k \cong F/F_k$ and ϕ_1 extends to a continuous map $\phi_1 : M_1 \to K(F/F_{k+1}, 1)$ inducing $G_1/(G_1)_{k+1} \cong F/F_{k+1}$. If (M_0, ϕ_0) is bordant to (M_1, ϕ_1) in $\Omega_3(F/F_k)$, then ϕ_0 also extends so that it induces $G_0/(G_0)_{k+1} \cong F/F_{k+1}$.*

Proof. Lemma 5.9 tells us there exists a bordism (W, Φ) between (M_0, ϕ_0) and (M_1, ϕ_1) over $K(F/F_k, 1)$ such that W contains only 2-handles with attaching circles in F_k and $\pi_1(W) \cong F/F_k$. Let $j_i : M_i \to W$ be inclusion maps so that $\Phi \circ j_i = \phi_i$, $i = 0, 1$.



Consider any collection $\{x_1, ..., x_k\} \in H^1(M_0)$ of cohomology classes. Then choose $y_i \in H^1(F/F_k)$ so that $\phi_0^*(y_i) = x_i$. Since $\pi_1(W) \cong F/F_k$ and $G_0/(G_0)_k \cong F/F_k$, Lemma 5.8 says that Massey products of length less than k vanish. Thus each of the following Massey products are uniquely defined:

$$\langle x_1, ..., x_k \rangle = \langle \phi_0^*(y_1), ..., \phi_0^*(y_k) \rangle = j_0^* \langle \Phi^*(y_1), ..., \Phi^*(y_k) \rangle .$$

If we can actually show that these Massey products vanish then we can use Lemma 5.8 to show that ϕ_0 also induces $G_0/(G_0)_{k+1} \cong F/F_{k+1}$, thus completing the proof. We will show $\langle \Phi^*(y_1), ..., \Phi^*(y_k)\rangle = 0$. Since $G_1/(G_1)_{k+1} \cong F/F_{k+1}$, Lemma 5.8 says Massey products for $H^1(M_1)$ of length less than $k+1$ vanish. In particular, length k Massey products are zero, thus

$$j_1^* \langle \Phi^*(y_1), ..., \Phi^*(y_k)\rangle = \langle \phi_1^*(y_1), ..., \phi_1^*(y_k)\rangle = 0.$$

Now consider the following short exact sequence

$$0 \longrightarrow H_2(M_1) \longrightarrow H_2(W) \longrightarrow H_2(W, M_1) \longrightarrow 0.$$

Since we can view W as $M_1 \times [0,1]$ with 2-handles attached along circles in F_k, we see that $H_2(W, M_1)$ is a free abelian group generated by the cores of the 2-handles (rel M_1). Thus this sequence splits and we can write $H_2(W) \cong H_2(M_1) \oplus H_2(W, M_1)$. Because the attaching circles of the 2-handles lie in F_k, the images of the generators of the latter summand are clearly spheres in $K(F/F_k, 1)$. But since $K(F/F_k, 1)$ has trivial higher homotopy groups, they must vanish in $H_2(F/F_k)$. Then by considering the dual splitting $H^2(W) \cong H^2(M_1) \oplus H^2(W, M_1)$ we know that the image of $H^2(F/F_k)$ must be contained in the summand $H^2(M_1)$ of $H^2(W)$. Therefore $j_1^* : H^2(W) \to H^2(M_1)$ must be injective on the image of $H^2(F/F_k)$, and we are able to conclude that $\langle \Phi^*(y_1), ..., \Phi^*(y_k)\rangle = 0$. □

Consider the following result of V. Turaev [Tu].

Lemma 5.11 (Turaev) *Let G be a finitely generated nilpotent group of nilpotency class at most $k \geq 1$, and let $\alpha \in H_3(G)$. Then there exists a closed, connected, oriented*

3-manifold M and a continuous map $\psi : M \to K(G,1)$ such that $\psi_*([M]) = \alpha$ and such that ψ induces an isomorphism $\pi_1(M)/(\pi_1(M))_k \cong G$ if and only if

(a) *the homomorphism on torsion subgroups* $\mathrm{Torsion}(H^2(G)) \to \mathrm{Torsion}(H_1(G))$ *defined by $x \mapsto x \cap \alpha$ is an isomorphism, and*

(b) *for any $h \in H_2(G)$, there exists $y \in H^1(G)$ such that*

$$h - (y \cap \alpha) \in \ker\left(H_2(G) \to H_2\left(\frac{G}{G_{k-1}}\right)\right).$$

Corollary 5.12 *For any bordism class $\alpha \in \Omega_3(F/F_k)$ there exists a closed, connected, oriented 3-manifold M and a continuous map $\psi : M \to K(F/F_k, 1)$ such that $(M, \psi) = \alpha$ in $\Omega_3(F/F_k)$ and ψ induces an isomorphism $\pi_1(M)/(\pi_1(M))_k \cong F/F_k$.*

Proof. We simply use the fact proved earlier that $\Omega_3(F/F_k) \cong H_3(F/F_k)$ and apply the lemma in the case that $G \cong F/F_k$. The group F/F_k is nilpotent with nilpotency $k-1$. The groups $H^2(F/F_k)$ and $H_1(F/F_k)$ are each torsion-free. Thus condition (a) of Lemma 5.11 is satisfied trivially. Using Stallings' exact sequence given in [St], we have the following commutative diagram

$$\begin{array}{ccccccc}
H_2(F) = 0 & \longrightarrow & H_2\left(\dfrac{F}{F_k}\right) & \xrightarrow{\cong} & \dfrac{F_k}{F_{k+1}} & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow 0 - map & & \\
H_2(F) = 0 & \longrightarrow & H_2\left(\dfrac{F}{F_{k-1}}\right) & \xrightarrow{\cong} & \dfrac{F_{k-1}}{F_k} & \longrightarrow & 0
\end{array}$$

which shows us that the map $H_2(F/F_k) \to H_2(F/F_{k-1})$ is the zero homomorphism. Thus condition (b) of Lemma 5.11 is also satisfied trivially. □

Lemma 5.13 *Let M be any closed, oriented 3-manifold with $\pi_1(M) = G$, and suppose there is a continuous map $\phi_k : M \to K(F/F_k, 1)$ inducing an isomorphism $G/G_k \cong F/F_k$ for some free group F. For $m \geq k$, (M, ϕ_k) is in the image of $(\pi_{m,k})_* : \Omega_3(F/F_m) \to \Omega_3(F/F_k)$ if and only if the isomorphism $G/G_k \cong F/F_k$ can be extended to an isomorphism $G/G_m \cong F/F_m$ induced by a continuous map $\phi_m : M \to K(F/F_m, 1)$ such that $(\pi_{m,k})_* (M, \phi_m) = (M, \phi_k)$.*

Proof. Suppose $(M, \phi_k) = (\pi_{m,k})_* (\alpha)$, for some $\alpha \in \Omega_3(F/F_m)$. By Corollary 5.12 there exists a closed, connected, oriented 3-manifold M' along with a continuous map $\psi : M' \to K(F/F_m, 1)$ that induces an isomorphism $\pi_1(M')/(\pi_1(M'))_m \cong F/F_m$ such that $(M', \psi) = \alpha$ in $\Omega_3(F/F_m)$. But $(M, \phi_k) = (\pi_{m,k})_* (\alpha) = (\pi_{m,k})_* (M', \psi) = (M', \pi_{m,k} \circ \psi)$, so (M, ϕ_k) and $(M', \pi_{m,k} \circ \psi)$ are bordant in $\Omega_3(F/F_k)$. In the case $m = k+1$, Lemma 5.10 gives the desired result. The case $m > k+1$ is achieved via induction. The converse is clear. □

We are now ready to continue our proof of Theorem 5.3. First, we are assuming that $\phi^{\gamma}_{f,k}$ exists, so Lemma 5.1 tells us that at the very least $f \in \mathcal{J}(k)$. We also assume that $\left(T^{\gamma}_f, \phi^{\gamma}_{f,k}\right)$ is trivial in $\Omega_3(F/F_k)$. In particular, $\left(T^{\gamma}_f, \phi^{\gamma}_{f,k}\right) = \left(T^{\gamma}_{id}, \phi^{\gamma}_{id,k}\right)$ in $\Omega_3(F/F_k)$. Also, we have

$$\frac{\pi_1(T^{\gamma}_{id})}{(\pi_1(T^{\gamma}_{id}))_m} \cong \frac{F}{F_m}, \text{ for all } m, \text{ and}$$

$$\frac{\pi_1(T^{\gamma}_f)}{\left(\pi_1(T^{\gamma}_f)\right)_m} \cong \frac{F}{F_m}, \text{ for all } m \leq k.$$

Then by Lemma 5.10 we can extend the latter isomorphism to

$$\frac{\pi_1(T_f^\gamma)}{\left(\pi_1(T_f^\gamma)\right)_{k+1}} \cong \frac{F}{F_{k+1}}.$$

By Lemma 5.1 we are able to conclude that $f \in \mathcal{J}(k+1)$ and that the continuous map $\phi_{f,k+1}^\gamma$ exists, allowing us to consider $\left(T_f^\gamma, \phi_{f,k+1}^\gamma\right) \in \Omega_3(F/F_{k+1})$. Moreover, since we are assuming that $\left(T_f^\gamma, \phi_{f,k}^\gamma\right)$ is trivial in $\Omega_3(F/F_k)$, we have

$$\left(T_f^\gamma, \phi_{f,k+1}^\gamma\right) \in \ker\left(\Omega_3\left(\frac{F}{F_{k+1}}\right) \stackrel{\left(\pi_{k+1,k}\right)_*}{\longrightarrow} \Omega_3\left(\frac{F}{F_k}\right)\right),$$

and by Corollary 5.7

$$\left(T_f^\gamma, \phi_{f,k+1}^\gamma\right) \in \text{Image}\left(\Omega_3\left(\frac{F}{F_{2k-1}}\right) \stackrel{\left(\pi_{2k-1,k+1}\right)_*}{\longrightarrow} \Omega_3\left(\frac{F}{F_{k+1}}\right)\right).$$

Thus Lemma 5.13 implies that the isomorphism

$$\frac{\pi_1(T_f^\gamma)}{\left(\pi_1(T_f^\gamma)\right)_{k+1}} \cong \frac{F}{F_{k+1}}$$

extends to an isomorphism

$$\frac{\pi_1(T_f^\gamma)}{\left(\pi_1(T_f^\gamma)\right)_{2k-1}} \cong \frac{F}{F_{2k-1}}.$$

Therefore, by Lemma 5.1, we are able to conclude that $f \in \mathcal{J}(2k-1)$. This completes the proof of Theorem 5.3. $\square$

5.3 Relating σ_k to the Johnson Homomorphism

The goal of this section is to describe how the homomorphism $\sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k)$ relates to Johnson's homomorphism

$$\tau_k : \mathcal{J}(k) \to D_k(H_1) \subset \operatorname{Hom}\left(H_1, \frac{F_k}{F_{k+1}}\right).$$

It turns out that τ_k factors through $\Omega_3(F/F_k)$. To see this, we will use Kitano's definition of τ_k in terms of Massey products, which we reviewed in Section 2.3.

Let $\mathfrak{X}$ denote the ring of formal power series in the noncommutative variables $t_1, ..., t_{2g}$, and let $\mathfrak{X}_k$ denote the submodule of $\mathfrak{X}$ corresponding to the degree k part. Because F_k/F_{k+1} is a submodule of $\mathfrak{X}_k$, we can consider the homomorphism

$$\tau_k : \mathcal{J}(k) \to \operatorname{Hom}(H_1, \mathfrak{X}_k)$$

defined in Theorem 2.1. Recall from Section 2.3 that we are considering the following dual bases:

$$\{x_1, ..., x_{2g}, y\} \in H_1(T_{f,1}),$$

$$\left\{x_1^*, ..., x_{2g}^*, y^*\right\} \in H^1(T_{f,1}), \text{ and}$$

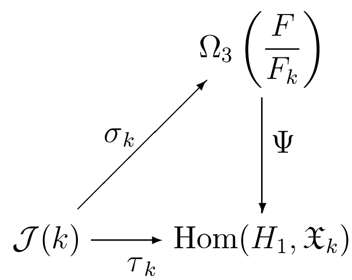
$$\{X_1, ..., X_{2g}\} \in H_2(T_{f,1}).$$

Define $\Psi' : \Omega_3(F/F_k, \zeta) \to \operatorname{Hom}(H_1, \mathfrak{X}_k)$ to be the map that sends the bordism class $\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right)$ to the homomorphism

$$x_i \longmapsto \sum_{j_1,\ldots,j_k} \left\langle \left\langle x_{j_1}^*, ..., x_{j_k}^* \right\rangle, X_i \right\rangle t_{j_1} \cdots t_{j_k}.$$

Let $i_* : \Omega_3(F/F_k) \to \Omega_3(F/F_k, \zeta)$ be the homomorphism induced by inclusion which sends $\left(T_f^\gamma, \phi_{f,k}^\gamma\right)$ to $\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right)$. Then we define $\Psi : \Omega_3(F/F_k) \to \mathrm{Hom}(H_1, \mathfrak{X}_k)$ to be the composition $\Psi = \Psi' \circ i_*$.

Theorem 5.14 *The map Ψ is a well-defined homomorphism. Moreover, the composition $\Psi \circ \sigma_k$ corresponds to the Johnson homomorphism τ_k so that we have the following commutative diagram.*



Proof. We only need to show that $\Psi' : \Omega_3(F/F_k, \zeta) \to \mathrm{Hom}(H_1, \mathfrak{X}_k)$ is a well-defined homomorphism, and the rest of the theorem clearly follows. We will need the following lemma.

Lemma 5.15 *Suppose (M_0, ϕ_0) and (M_1, ϕ_1) are closed, oriented 3-manifolds with $\pi_1(M_i) = G_i$ and continuous maps $\phi_i : M_i \to K(G_0/(G_0)_k, 1)$. Further suppose ϕ_1 induces an isomorphism $G_1/(G_1)_k \cong G_0/(G_0)_k$. If (M_0, ϕ_0) is bordant to (M_1, ϕ_1) in $\Omega_3(G_0/(G_0)_k)$ and all Massey products for $H^1(M_0)$ of length less than k vanish, then $\phi = (\phi_0)_*^{-1} \circ (\phi_1)_* : H_1(M_1) \to H_1(M_0)$ is an isomorphism such that for $x_i \in H^1(M_0)$, $\mathcal{E}_i \in H_2(M_0)$ Poincaré dual to x_i, and $\mathcal{F}_i \in H_2(M_1)$ Poincaré dual to $\phi^*(x_i) \in H^1(M_1)$ we have*

$$\left\langle \langle x_{j_1}, ..., x_{j_k} \rangle, \mathcal{E}_i \right\rangle = \left\langle \langle \phi^*(x_{j_1}), ..., \phi^*(x_{j_k}) \rangle, \mathcal{F}_i \right\rangle$$

where $\langle\ \ ,\ \ \rangle$ is the dual pairing of $H^2(M_i)$ and $H_2(M_i)$.

Proof. Since (M_0, ϕ_0) is bordant to (M_1, ϕ_1) in $\Omega_3(G_0/(G_0)_k)$, we must also have $(\phi_0)_*([M_0]) = (\phi_1)_*([M_1])$ in $H_3(G_0/(G_0)_k)$ where $[M_i]$ is the fundamental class in $H_3(M_i)$. The bordism (W, Φ) between (M_0, ϕ_0) and (M_1, ϕ_1) can be chosen so that Φ induces an isomorphism $\pi_1(W) \cong G_0/(G_0)_k$ and the inclusion maps $j_i : M_i \to W$ induce isomorphisms

$$\frac{G_i}{(G_i)_k} \cong \frac{\pi_1(W)}{(\pi_1(W))_k}.$$

W. Dwyer proves in [Dw] (Corollary 2.5) that for cohomology classes $\alpha_i \in H^1(W)$ we have $\langle \alpha_1, ..., \alpha_m \rangle = 0$ if and only if $j_0^* \langle \alpha_1, ..., \alpha_m \rangle = 0$ for $m < k$. However, by naturality of Massey products given in (2.2.2), $j_0^* \langle \alpha_1, ..., \alpha_m \rangle \subset \langle j_0^*(\alpha_1), ..., j_0^*(\alpha_m) \rangle$, and the latter is 0 since Massey products of length less than k vanish for $H^1(M_0)$. Thus $\langle \alpha_1, ..., \alpha_m \rangle = 0$ for all $\alpha_i \in H^1(W)$. Moreover, $j_1^* : H^1(W) \to H^1(M_1)$ is an isomorphism. Then for any $y_i \in H^1(M_1)$ there exists an $\alpha_i \in H^1(W)$ such that $j_1^*(\alpha_i) = y_i$. Thus for $m < k$ we have

$$\begin{aligned} \langle y_1, ..., y_m \rangle &= \langle j_1^*(\alpha_1), ..., j_1^*(\alpha_m) \rangle \\ &= j_1^* \langle \alpha_1, ..., \alpha_m \rangle \\ &= 0 \end{aligned}$$

where the second equality follows from naturality. So then we have shown that all Massey products of length less than k vanish also for $H^1(W)$ and $H^1(M_1)$. Thus Massey products for $H^1(M_0)$, $H^1(M_1)$, and $H^1(W)$ of length k are uniquely defined.

Consider $x_i \in H^1(M_0)$ with Poincaré dual $\mathcal{E}_i \in H_2(M_0)$. Let $\mathcal{F}_i \in H_2(M_1)$ be Poincaré dual to $\phi^*(x_i) \in H^1(M_1)$, where ϕ is the isomorphism given by the composition $\phi = (\phi_0)_*^{-1} \circ (\phi_1)_* : H_1(M_1) \to H_1(M_0)$. Then we have

$$\begin{aligned}
(\phi_0)_* (\mathcal{E}_i) &= (\phi_0)_* (x_i \cap [M_0]) \\
&= (\phi_0^*)^{-1} (x_i) \cap (\phi_0)_* ([M_0]) \\
&= \left((\phi_1^*)^{-1} \circ \phi^*\right) (x_i) \cap (\phi_1)_* ([M_1]) \\
&= (\phi_1)_* (\phi^*(x_i) \cap [M_1]) \\
&= (\phi_1)_* (\mathcal{F}_i),
\end{aligned}$$

where the second and fourth equalities follow from the naturality of cap products.

Now choose $\beta_i \in H^1(G_0/(G_0)_k)$ such that $\phi_0^*(\beta_i) = x_i$. Then

$$\begin{aligned}
\langle\langle x_{j_1}, ..., x_{j_k}\rangle, \mathcal{E}_i\rangle &= \left\langle\left\langle \phi_0^*(\beta_{j_1}), ..., \phi_0^*(\beta_{j_k})\right\rangle, \mathcal{E}_i\right\rangle \\
&= \left\langle\left\langle \beta_{j_1}, ..., \beta_{j_k}\right\rangle, (\phi_0)_* (\mathcal{E}_i)\right\rangle \\
&= \left\langle\left\langle \beta_{j_1}, ..., \beta_{j_k}\right\rangle, (\phi_1)_* (\mathcal{F}_i)\right\rangle \\
&= \left\langle\left\langle \phi_1^*(\beta_{j_1}), ..., \phi_1^*(\beta_{j_k})\right\rangle, \mathcal{F}_i\right\rangle \\
&= \left\langle\left\langle (\phi^* \circ \phi_0^*) (\beta_{j_1}), ..., (\phi^* \circ \phi_0^*) (\beta_{j_k})\right\rangle, \mathcal{F}_i\right\rangle \\
&= \langle\langle \phi^*(x_{j_1}), ..., \phi^*(x_{j_k})\rangle, \mathcal{F}_i\rangle .
\end{aligned}$$

This completes the proof of the lemma. □

Consider the mapping classes $f, h \in \mathcal{J}(k)$. We have the dual bases mentioned above for specific homology and cohomology groups of $T_{f,1}$. Consider the following dual bases defined in the same manner for $T_{h,1}$:

$$\{w_1, ..., w_{2g}, z\} \in H_1(T_{h,1}),$$

$$\left\{w_1^*, ..., w_{2g}^*, z^*\right\} \in H^1(T_{h,1}), \text{ and}$$

$$\{W_1, ..., W_{2g}\} \in H_2(T_{h,1}).$$

Recall that T_f^γ was constructed from $T_{f,1}$ by filling the boundary $\partial T_{f,1} = \partial\Sigma_{g,1} \times S^1$ with a solid torus $\partial\Sigma_{g,1} \times D^2$. Let $\psi_f : T_{f,1} \to T_f^\gamma$ be the inclusion map, and then we have a basis

$$\left\{a_1^*, ..., a_{2g}^*\right\} \in H^1(T_f^\gamma)$$

where $a_i^* = \left(\left(\psi_f\right)_* (x_i)\right)^*$. Since x_i^* is the Hom dual of x_i, by definition we have that the dual pairing is $\langle x_i^*, x_j\rangle = \delta_{ij}$. Similarly a_i^* is the Hom dual of $\left(\psi_f\right)_* (x_i)$, and thus $\left\langle \psi_f^*(a_i^*), x_j\right\rangle = \left\langle a_i^*, \left(\psi_f\right)_* (x_i)\right\rangle = \delta_{ij}$. Note that this implies that $\psi_f^*(a_i^*) = x_i^*$. Letting A_i denote the Poincaré dual of a_i^* gives a basis for $H_2(T_f^\gamma)$:

$$\{A_1, ..., A_{2g}\} \in H_2(T_f^\gamma).$$

By carefully examining the following commutative diagram, we see $\left(\psi_f\right)_* (X_i) = A_i$.

$$\begin{array}{ccccc}
H_1\left(T_{f,1}, \partial T_{f,1}\right) & \xrightarrow[\text{Hom } dual]{\cong} & H^1\left(T_{f,1}, \partial T_{f,1}\right) & \xrightarrow{\cap\left[T_{f,1}, \partial T_{f,1}\right]} & H_2\left(T_{f,1}\right) \\
\uparrow j_* & & \downarrow j^* & & \\
H_1\left(T_{f,1}\right) & \xrightarrow[\text{Hom } dual]{\cong} & H^1\left(T_{f,1}\right) & & \downarrow \left(\psi_f\right)_* \\
\downarrow \left(\psi_f\right)_* & & \uparrow \psi_f^* & & \\
H_1\left(T_f^\gamma\right) & \xrightarrow[\text{Hom } dual]{\cong} & H^1\left(T_f^\gamma\right) & \xrightarrow{\cap\left[T_f^\gamma\right]} & H_2\left(T_f^\gamma\right)
\end{array}$$

Finally, let $\bar{A}_i \in H_2(T_h^\gamma)$ denote the Poincaré dual to $\phi^*(a_i^*)$, where ϕ is the isomorphism guaranteed by the following corollary. Then for $T_{h,1}$ we similarly have $\psi_h^*(\phi^*(a_i^*)) = w_i^*$ and $(\psi_h)_*(W_i) = \bar{A}_i$. We have the following immediate corollary to Lemma 5.15.

Corollary 5.16 *If* $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) = \left(T_h^\gamma, \phi_{h,k}^\gamma\right)$ *in* $\Omega_3(F/F_k)$*, then*

$$\phi = (\phi_0)_*^{-1} \circ (\phi_1)_* : H_1(T_g^\gamma) \to H_1(T_f^\gamma)$$

is an isomorphism such that

$$\left\langle \left\langle a_{j_1}^*, ..., a_{j_k}^* \right\rangle, A_i \right\rangle = \left\langle \left\langle \phi^*(a_{j_1}^*), ..., \phi^*(a_{j_k}^*) \right\rangle, \bar{A}_i \right\rangle$$

where $\bar{A}_i$ *is Poincaré dual to* $\phi^*(a_i^*)$*.*

Lemma 5.17 *If* $\left(T_{f,1}, \partial T_{f,1}, \phi_{f,k}\right) = \left(T_{h,1}, \partial T_{h,1}, \phi_{h,k}\right)$ *in* $\Omega_3(F/F_k, \zeta)$*, then*

$$\left\langle \left\langle x_{j_1}^*, ..., x_{j_k}^* \right\rangle, X_i \right\rangle = \left\langle \left\langle w_{j_1}^*, ..., w_{j_k}^* \right\rangle, W_i \right\rangle.$$

Proof. Since $f, h \in \mathcal{J}(k)$, Theorem 2.2 says that the Massey products for $(T_{f,1}, \partial T_{f,1})$ and $(T_{h,1}, \partial T_{h,1})$ of length less than k vanish. Thus $\left\langle x_{j_1}^*, ..., x_{j_k}^* \right\rangle$ and $\left\langle w_{j_1}^*, ..., w_{j_k}^* \right\rangle$ are uniquely defined. By Corollary 5.5, we know that $\left(T_f^\gamma, \phi_{f,k}^\gamma\right) = \left(T_h^\gamma, \phi_{h,k}^\gamma\right)$ in $\Omega_3(F/F_k)$.

So we let ϕ be the isomorphism guaranteed by Corollary 5.16. Then we have

$$\begin{aligned}
\left\langle \left\langle x^*_{j_1}, ..., x^*_{j_k} \right\rangle, X_i \right\rangle &= \left\langle \left\langle \psi^*_f(a^*_{j_1}), ..., \psi^*_f(a^*_{j_k}) \right\rangle, X_i \right\rangle \\
&= \left\langle \left\langle a^*_{j_1}, ..., a^*_{j_k} \right\rangle, \left(\psi_f\right)_* (X_i) \right\rangle \\
&= \left\langle \left\langle a^*_{j_1}, ..., a^*_{j_k} \right\rangle, A_i \right\rangle \\
&= \left\langle \left\langle \phi^*(a^*_{j_1}), ..., \phi^*(a^*_{j_k}) \right\rangle, \bar{A}_i \right\rangle \\
&= \left\langle \left\langle \phi^*(a^*_{j_1}), ..., \phi^*(a^*_{j_k}) \right\rangle, (\psi_h)_* (W_i) \right\rangle \\
&= \left\langle \psi^*_h \left\langle \phi^*(a^*_{j_1}), ..., \phi^*(a^*_{j_k}) \right\rangle, W_i \right\rangle \\
&= \left\langle \left\langle \psi^*_h(\phi^*(a^*_{j_1})), ..., \psi^*_h(\phi^*(a^*_{j_k})) \right\rangle, W_i \right\rangle \\
&= \left\langle \left\langle w^*_{j_1}, ..., w^*_{j_k} \right\rangle, W_i \right\rangle.
\end{aligned}$$

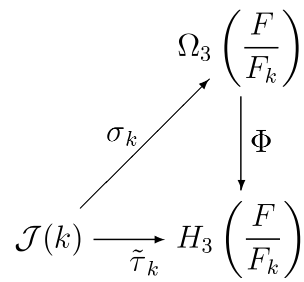
□

This proves that $\Psi' : \Omega_3(F/F_k, \zeta) \to \mathrm{Hom}(H_1, \mathfrak{X}_k)$ is a well-defined homomorphism and completes the proof of Theorem 5.14. □

5.4 Relating σ_k to Morita's Homomorphism

As we have already seen in the proof of Corollary 5.7, there exists an isomorphism $\Phi : \Omega_3(F/F_k) \to H_3(F/F_k)$ given by $(M, \phi) \mapsto \phi_*([M])$, where $[M]$ is the fundamental class in $H_3(M)$. Because of this, one may guess that there is a relationship between $\sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k)$ and Morita's refinement $\tilde{\tau}_k : \mathcal{J}(k) \to H_3(F/F_k)$ discussed in Section 2.4. This assumption turns out to be correct, and the two homomorphisms are in fact equivalent. However, σ_k gives a representation that is much more geometric, and as we will see in Chapter 6, σ_k leads to interesting questions that $\tilde{\tau}_k$ does not.

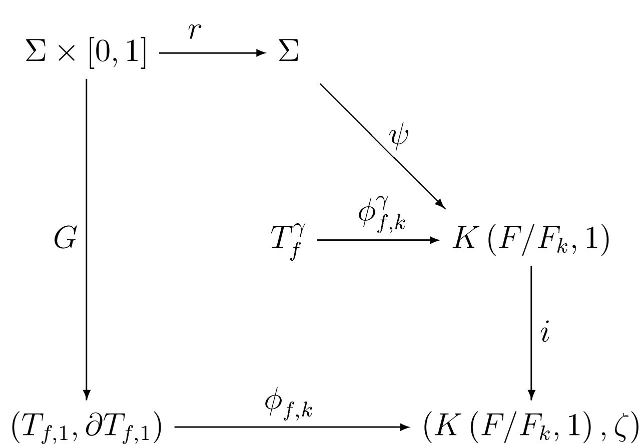
Theorem 5.18 *The homomorphism $\sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k)$ coincides with Morita's refinement of the Johnson homomorphism so that we have a commutative diagram.*



Corollary 5.19 *The kernel of Morita's refinement $\tilde{\tau}_k$ is $\mathcal{J}(2k-1)$.*

Proof. This is an immediate consequence of Theorem 5.18 and Corollary 5.4. □

Proof of Theorem 5.18. Consider a genus g punctured surface $\Sigma = \Sigma_{g,1}$ and $f \in \mathcal{J}(k)$. Let $r : \Sigma \times [0,1] \to \Sigma$ be a retraction, $\psi : \Sigma \to K(F/F_k, 1)$ be a continuous map that induces the canonical epimorphism $F \twoheadrightarrow F/F_k$, and $i : K(F/F_k, 1) \to (K(F/F_k, 1), \zeta)$ be the inclusion map. Also let $G : \Sigma \times [0,1] \to (T_{f,1}, \partial T_{f,1})$ be the composition of the "gluing map" $\Sigma \times [0,1] \to T_{f,1}$ and the inclusion $T_{f,1} \to (T_{f,1}, \partial T_{f,1})$. Recall that the maps $\phi_{f,k}$ and $\phi^\gamma_{f,k}$ defined at the beginning of Section 5.2 are defined only up to homotopy. We choose them so that the following diagram commutes.



That is, we have $\phi_{f,k} \circ G = i \circ \psi \circ r$.

Consider the fundamental class $[T_{f,1}, \partial T_{f,1}] \in H_3(T_{f,1}, \partial T_{f,1})$, and suppose that $(t_f, \partial t_f) \in C_3(T_{f,1}, \partial T_{f,1})$ is a corresponding relative 3-cycle. Now we choose a 2-chain $\sigma \in C_2(\Sigma \times [0,1])$ so that $\partial\sigma$ is in the homotopy class of a simple closed curve on $\Sigma \times \{0\}$ parallel to the boundary $\partial\Sigma \times \{0\}$. Let σ also denote $r_\#(\sigma) \in C_2(\Sigma)$, and choose a 3-chain $\rho \in C_3(\Sigma \times [0,1])$ so that $G_\#(\rho) = (t_f, \partial t_f)$ and $\partial\rho = \sigma - f_\#(\sigma) + (\partial\sigma \times [0,1])$.

Consider the restriction $r|_{\partial\Sigma \times [0,1]}$. Then $r_\#(\partial\sigma \times [0,1]) = \varepsilon \in C_2(\partial\Sigma)$, and

$$\begin{aligned}
\partial r_\#(\rho) &= r_\# \partial(\rho) \\
&= r_\#(\sigma - f_\#(\sigma) + (\partial\sigma \times [0,1])) \\
&= r_\#(\sigma) - r_\#(f_\#(\sigma)) + r_\#(\partial\sigma \times [0,1]) \\
&= \sigma - f_\#(\sigma) + \varepsilon
\end{aligned}$$

Since f is the identity on the boundary, we must have $\partial\sigma - f_\#(\partial\sigma) = 0$, and therefore $0 = \partial(\partial r_\#(\rho)) = \partial(\sigma - f_\#(\sigma) + \varepsilon) = \partial\sigma - f_\#(\partial\sigma) + \partial\varepsilon = \partial\varepsilon$. Since $H_2(\partial\Sigma)$ is trivial, there must be a 3-chain $\eta \in C_3(\partial\Sigma)$ such that $\partial\eta = \varepsilon$. Let $j : \Sigma \to \Sigma \times [0,1]$ be the inclusion map, and consider $j_\#(\eta) \in C_3(\partial\Sigma \times [0,1]) \to C_3(\Sigma \times [0,1])$. Define $c_f \in C_3(\Sigma)$ to be

$$\begin{aligned}
c_f &= r_\#(\rho - j_\#(\eta)) \\
&= r_\#(\rho) - r_\# j_\#(\eta) \\
&= r_\#(\rho) - \eta.
\end{aligned}$$

Then $\partial c_f = \partial r_\#(\rho) - \partial\eta = (\sigma - f_\#(\sigma) + \varepsilon) - \varepsilon = \sigma - f_\#(\sigma)$.

Also, since $j_{\#}(\eta) \in C_3(\Sigma \times [0,1])$ is carried by the subcomplex $\partial\Sigma \times [0,1]$, $G_{\#}(j_{\#}(\eta))$ must be carried by $\partial T_{f,1}$. Thus $G_{\#}(j_{\#}(\eta)) = 0$, and

$$\begin{aligned} G_{\#}(\rho - j_{\#}(\eta)) &= G_{\#}(\rho) - G_{\#}(j_{\#}(\eta)) \\ &= (t_f, \partial t_f)\,. \end{aligned}$$

Let $\bar{c}_f = \psi_{\#}(c_f) \in C_3(F/F_k)$. Then $\bar{c}_f$ is a 3-cycle since $f \in \mathcal{J}(k)$ induces the identity on F/F_k. Let $[\bar{c}_f] \in H_3(F/F_k)$ denote the corresponding homology class, and

$$\begin{aligned} i_*([\bar{c}_f]) &= [i_{\#}(\bar{c}_f)] \\ &= \left[(i \circ \psi)_{\#}(c_f)\right] \\ &= \left[(i \circ \psi \circ r)_{\#}(\rho - j_{\#}(\eta))\right] \\ &= \left[\left(\phi_{f,k} \circ G\right)_{\#}(\rho - j_{\#}(\eta))\right] \\ &= \left[\left(\phi_{f,k}\right)_{\#}(t_f, \partial t_f)\right] \\ &= \left(\phi_{f,k}\right)_*([T_{f,1}, \partial T_{f,1}]) \end{aligned}$$

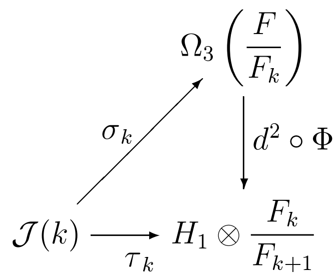
On the other hand, we also have $i_*((\phi^{\gamma}_{f,k})_*([T^{\gamma}_f])) = \left(\phi_{f,k}\right)_*([T_{f,1}, \partial T_{f,1}])$, and since $i_* : H_3(F/F_k) \to H_3(F/F_k, \zeta)$ is an isomorphism, we must have $[\bar{c}_f] = (\phi^{\gamma}_{f,k})_*([T^{\gamma}_f])$.

Finally, notice that our choices of $\sigma \in C_2(\Sigma)$ and $c_f \in C_3(\Sigma)$ certainly qualify as choices for $\sigma \in C_2(F)$ and $c_f \in C_3(F)$, respectively, in the construction of Morita's homomorphism in Section 2.4. Thus we have $(\Phi \circ \sigma_k)(f) = \Phi(T^{\gamma}_f, \phi^{\gamma}_{f,k}) = (\phi^{\gamma}_{f,k})_*([T^{\gamma}_f]) = [\bar{c}_f] = \tilde{\tau}_k(f)$, and the theorem is proved. □

Now that we see that $\sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k)$ and Morita's homomorphism are indeed equivalent, we can describe in a different way how σ_k relates to Johnson's homomorphism $\tau_k : \mathcal{J}(k) \to H_1 \otimes F_k/F_{k+1}$. Recall the differential

$$d^2 : H_3\left(\frac{F}{F_k}\right) \to H_1 \otimes \frac{F_k}{F_{k+1}}$$

discussed in Section 2.4. Then τ_k factors through $\Omega_3(F/F_k)$ so that the following diagram commutes.



Chapter 6

A Spin Bordism Representation of the Mapping Class Group

We introduced in Chapter 5 a new representation $\sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k)$ which we then showed was equivalent to Morita's homomorphism $\tilde{\tau}_k : \mathcal{J}(k) \to H_3(F/F_k)$. Because of the range of the latter homomorphism, it may seem preferable to the reader. However, σ_k has its advantages. First, it simply has a much more geometric nature to it. Second, and perhaps most importantly, it naturally leads to an interesting question that $\tilde{\tau}_k$ does not. What happens when we add more structure to the bordism group? More specifically, what is the result of replacing the bordism group $\Omega_3(F/F_k)$ with the spin bordism group $\Omega_3^{spin}(F/F_k)$?

6.1 A Spin Bordism Invariant of $\mathcal{J}(k)$

Recall that a spin structure can be thought of as a trivialization of the stable tangent bundle restricted to the 2-skeleton, and every oriented 3-manifold has a spin structure.

Since a spin structure on a manifold induces a spin structure on its boundary, we can define the *3-dimensional spin bordism group* $\Omega_3^{spin}(X)$ in exactly the same way as the oriented bordism group $\Omega_3(X)$ with the additional requirement that spin structures on spin bordant 3-manifolds must extend to a spin structure on the 4-dimensional bordism between them. That is, elements of $\Omega_3^{spin}(X)$ are equivalence classes of triples (M, ϕ, σ) consisting of a closed, spin 3-manifold M with spin structure σ and a continuous map $\phi : M \to X$. We say two elements (M_0, ϕ_0, σ_0) and (M_1, ϕ_1, σ_1) are equivalent, or *spin bordant over* X, if there is a triple (W, Φ, σ) consisting of a compact, spin 4-manifold (W, σ) with boundary $\partial(W, \sigma) = (M_0, \sigma_0) \amalg -(M_1, \sigma_1)$ and a continuous map $\Phi : W \to X$ satisfying $\Phi|_{M_i} = \phi_i$.

Further recall that the spin structures for a spin manifold M are enumerated by $H^1(M; \mathbb{Z}_2)$. Thus, for example, the number of possible spin structures on a punctured, oriented surface $\Sigma_{g,1}$ of genus g is $|H^1(\Sigma_{g,1}; \mathbb{Z}_2)| = \left|\mathbb{Z}_2^{2g}\right| = 2^{2g}$. If we fix a spin structure on $\Sigma_{g,1}$, then we can extend it to the product $\Sigma_{g,1} \times [0,1]$. Now consider the mapping class $f \in \mathcal{J}(k)$ for $\Sigma_{g,1}$. For $k \geq 2$, f acts trivially on $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$ and on the set of spin structures. Thus the spin structure on $\Sigma_{g,1} \times [0,1]$ can be extended to the mapping torus $T_{f,1}$. The number of possible spin structures for $T_{f,1}$ is $|H^1(T_{f,1}; \mathbb{Z}_2)| = \left|\mathbb{Z}_2^{2g+1}\right| = 2^{2g+1}$, where the extra factor of 2 corresponds to the extra generator $\gamma \in \pi_1(T_{f,1})$. Remember that we construct T_f^γ from $T_{f,1}$ by performing a Dehn filling along γ, i.e. filling the boundary $\partial T_{f,1} = \partial\Sigma_{g,1} \times S^1$ with $\partial\Sigma_{g,1} \times D^2$. Then, as long as we choose the spin structure for γ which extends over a disk, we can extend the spin structure on $T_{f,1}$ to a spin structure σ on T_f^γ. Again, the number of possible spin structures for T_f^γ is $\left|H^1(T_f^\gamma; \mathbb{Z}_2)\right| = \left|\mathbb{Z}_2^{2g}\right| = 2^{2g}$, and these exactly correspond to the spin structures on $\Sigma_{g,1}$.

Theorem 6.1 *Let $\Sigma_{g,1}$ be a punctured surface of genus g with a fixed spin structure. Let σ denote the corresponding spin structure on T_f^γ for all $f \in \mathcal{J}(k)$, $k \geq 2$. Then there is a well-defined homomorphism*

$$\eta_k : \mathcal{J}(k) \to \Omega_3^{spin}\left(\frac{F}{F_k}\right)$$

defined by $\eta_k(f) = \left(T_f^\gamma, \phi_{f,k}^\gamma, \sigma\right)$.

Proof. This follows directly from the proof that $\sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k)$ is a well-defined homomorphism since the spin structure on $T_f^\gamma \amalg T_h^\gamma$ naturally extends over the product $\left(T_f^\gamma \amalg T_h^\gamma\right) \times [0,1]$ and the spin structure on $\Sigma_{g,1}$ naturally extends over the product $\Sigma_{g,1} \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]$. □

First, we point out that if we compose this homomorphism with a "forgetful" map which ignores the spin structure then we obtain our original homomorphism σ_k. Second, recall the proof of Corollary 5.7 where we pointed out that, by using the Atiyah-Hirzebruch spectral sequence, one could build the n-dimensional bordism group $\Omega_n(X, A)$ using $H_p(X, A; \Omega_q)$ with $p + q = n$ as building blocks. In the same way, the n-dimensional spin bordism group $\Omega_n^{spin}(X, A)$ is constructed out of $H_p(X, A; \Omega_q^{spin})$ with $p + q = n$, where $\Omega_q^{spin} = \Omega_q^{spin}(\cdot)$ is the spin bordism group of a single point. Unlike the previous case for $n = 3$, all but one of the coefficient groups are nontrivial. In particular, since $\Omega_0^{spin} \cong \mathbb{Z}$, $\Omega_1^{spin} \cong \Omega_2^{spin} \cong \mathbb{Z}_2$, and $\Omega_3^{spin} \cong \{e\}$, we have that $\Omega_3^{spin}(F/F_k)$ is built out of

$$\begin{array}{rcccc} H_3(F/F_k;\Omega_0^{spin}) & \cong & H_3(F/F_k) & \cong & \Omega_3(F/F_k), \\ H_2(F/F_k;\Omega_1^{spin}) & \cong & H_2(F/F_k)\otimes\mathbb{Z}_2 & \cong & F_k/F_{k+1}\otimes\mathbb{Z}_2, \\ H_1(F/F_k;\Omega_2^{spin}) & \cong & H_1(F/F_k)\otimes\mathbb{Z}_2 & \cong & \mathbb{Z}_2^{2g}, \text{ and} \\ H_0(F/F_k;\Omega_3^{spin}) & \cong & 0. & & \end{array}$$

And so at the very least we see that there is potential for η_k to give much more information about the structure of the group $\mathcal{J}(k)$.

6.2 A Closer Look at η_2

We will now investigate the specific case when $k = 2$ and see what information $\eta_2 : \mathcal{J}(2) \to \Omega_3^{spin}(F/F_2)$ gives us about the Torelli group $\mathcal{J}(2)$. We have already seen that the original Johnson homomorphism τ_2 factors through $\Omega_3^{spin}(F/F_2)$ (see Theorem 5.14.) In this section we will see that, in fact, the Birman-Craggs homomorphisms $\{\rho_q\}$ also factor through $\Omega_3^{spin}(F/F_2)$. Therefore, this new homomorphism η_2 in a certain sense combines the Johnson homomorphism and the Birman-Craggs homomorphisms into a single homomorphism.

Consider any mapping class $f \in \mathcal{J}(2)$ and fix a spin structure on $\Sigma_{g,1}$. Let σ be the corresponding spin structure on T_f^γ. Finally let $\phi_f^\gamma = \phi_{f,2}^\gamma : T_f^\gamma \to K(F/F_2, 1)$ be a continuous map which induces the canonical epimorphism $\pi_1(T_f^\gamma) \twoheadrightarrow F/F_2 \cong \mathbb{Z}^{2g}$. Then the image under η_2 of f is $\left(T_f^\gamma, \phi_f^\gamma, \sigma\right)$.

Let $\alpha \in H^1(T_f^\gamma;\mathbb{Z})$ denote a primitive cohomology class. The group $\left[T_f^\gamma, S^1\right]$ of homotopy classes of maps $T_f^\gamma \to S^1$ is in one-to-one correspondence with $\mathrm{Hom}(\pi_1(T_f^\gamma),\mathbb{Z})$. In fact, there is an isomorphism $\left[T_f^\gamma, S^1\right] \cong H^1(T_f^\gamma;\mathbb{Z})$. So there is a continuous map

$\psi_\alpha : T_f^\gamma \to S^1$ corresponding to $\alpha \in H^1(T_f^\gamma;\mathbb{Z})$. There is a connected surface S embedded in T_f^γ which represents a class in $H_2(T_f^\gamma)$ Poincaré dual to α, and this surface S represents the same homology class in $H_2(T_f^\gamma)$ as $\psi_\alpha^{-1}(p)$ does, where $p \in S^1$ is a regular value of ψ_α. (If $p \in S^1$ is a regular value of ψ_α, then $\psi_\alpha^{-1}(p)$ is an embedded, codimension 1 submanifold of T_f^γ. That is, $\psi_\alpha^{-1}(p)$ is an embedded surface in T_f^γ.)

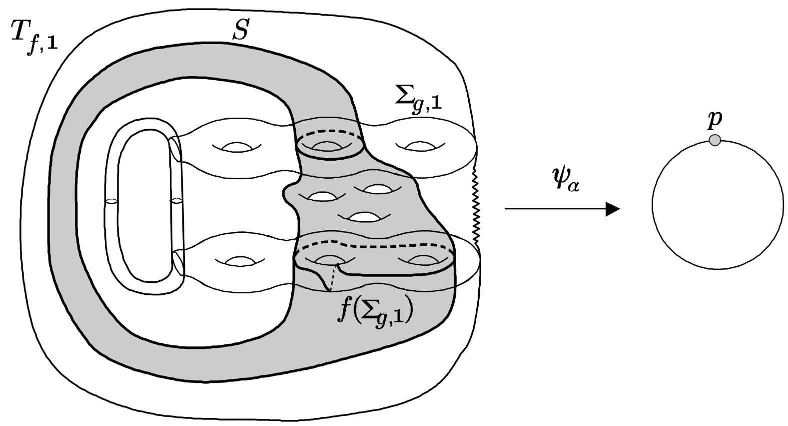


Figure 6.1: Embedding of S into $T_{f,1} \hookrightarrow T_f^\gamma$ and the map ψ_α

Let $\pi_\alpha : K(F/F_2, 1) \to S^1$ be a continuous map such that ψ_α is homotopic to $\pi_\alpha \circ \phi_f^\gamma$, and let $(\pi_\alpha)_* : \Omega_3^{spin}(F/F_2) \to \Omega_3^{spin}(S^1)$ denote the induced bordism homomorphism. Then we can define a homomorphism

$$\omega_\alpha = (\pi_\alpha)_* \circ \eta_2 : \mathcal{J}(2) \to \Omega_3^{spin}(S^1)$$

by sending $f \in \mathcal{J}(2)$ to the bordism class $\left(T_f^\gamma, \psi_\alpha, \sigma\right) \in \Omega_3^{spin}(S^1)$. Again, using the Atiyah-Hirzebruch spectral sequence, we see that $\Omega_3^{spin}(S^1) \cong \Omega_2^{spin} \cong \mathbb{Z}_2$. The specific isomorphism is given by $(M, \phi, \sigma) \mapsto \left(\phi^{-1}(p), \sigma|_{\phi^{-1}(p)}\right)$, where $p \in S^1$ is a regular value of ϕ. We can see by this isomorphism that the spin structure on T_f^γ restricts to a spin structure on $S = \psi_\alpha^{-1}(p)$.

Theorem 6.2 *The fixed spin structure on $\Sigma_{g,1}$ has a canonically associated quadratic form $q : H_1(\Sigma_{g,1};\mathbb{Z}_2) \to \mathbb{Z}_2$. If $\mathrm{Arf}(\Sigma_{g,1}, q) = 0$, then there is a primitive cohomology class $\alpha \in H^1(T_f^\gamma;\mathbb{Z})$ such that the homomorphisms $\omega_\alpha : \mathcal{J}(2) \to \Omega_3^{spin}(S^1)$ is equivalent to the Birman-Craggs homomorphism $\rho_q : \mathcal{J}(2) \to \mathbb{Z}_2$.*

We note that the hypothesis $\mathrm{Arf}(\Sigma_{g,1}, q) = 0$ is necessary for the Birman-Craggs homomorphism $\rho_q : \mathcal{J}(2) \to \mathbb{Z}_2$ to be defined. See Chapter 3 for details.

We have a surface $S = \psi_\alpha^{-1}(p)$ embedded in T_f^γ. To determine whether the image of f under the homomorphism $\omega_\alpha : \mathcal{J}(2) \to \Omega_3^{spin}(S^1)$ is trivial or not, we simply need to determine $(S, \sigma|_S) \in \Omega_2^{spin}$. However, this is just the well-known Arf invariant of S with respect to $\sigma|_S$. We defined the Arf invariant $\mathrm{Arf}(\Sigma, q)$ for a closed surface Σ and $\mathbb{Z}_2$-quadratic form $q : H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$ in Chapter 3. Now for the spin structure $\sigma|_S$ on S let $q_\sigma : H_1(S;\mathbb{Z}_2) \to \mathbb{Z}_2$ be the corresponding $\mathbb{Z}_2$-quadratic form. Namely, q_σ is defined to be the quadratic form given by $q_\sigma(x) = 0$ if $\sigma|_x$ is the spin structure that extends over a disk and $q_\sigma(x) = 1$ if $\sigma|_x$ does not extend over a disk. It is the work of Johnson in [J1] that tells us this quadratic form is equivalent to the quadratic form discussed in Chapter 3. Then we have

$$\mathrm{Arf}(S, q_\sigma) = \mathrm{Arf}(S, \sigma|_S) = (S, \sigma|_S) \in \Omega_2^{spin}.$$

We will also need a more general definition of the Arf invariant which includes surfaces with boundary. The definition is the same except for a small change to the $\mathbb{Z}_2$-quadratic form q. In particular we have a $\mathbb{Z}_2$-quadratic form

$$q : \frac{H_1(\Sigma;\mathbb{Z}_2)}{i_*(H_1(\partial\Sigma;\mathbb{Z}_2))} \to \mathbb{Z}_2$$

where i_* is induced by inclusion $i : \partial\Sigma \to \Sigma$. Then for a symplectic basis $\{x_i, y_i\}$ of the quotient $H_1(\Sigma;\mathbb{Z}_2)/i_*(H_1(\partial\Sigma;\mathbb{Z}_2))$, the *Arf invariant* of Σ with respect to q is defined to be

$$\mathrm{Arf}(\Sigma, q) = \sum_{i=1}^{g} q(x_i)q(y_i) \pmod 2.$$

Notice that if the surface Σ happens to be embedded in S^3 then this definition is the same as the definition of the *Arf invariant* $\mathrm{Arf}(L)$ *of an oriented link* L in S^3 with components $\{L_i\}$ and satisfying the property that the linking number is $\mathrm{lk}\,(L_i, L - L_i) \equiv 0$ modulo 2. The surface Σ would be a Seifert surface for the link, and q would be the mod 2 Seifert self-linking form on $H_1(\Sigma;\mathbb{Z}_2)/i_*(H_1(\partial\Sigma;\mathbb{Z}_2))$, where the self-linking is computed with respect to a push-off in a direction normal to the surface. See the W. Lickorish text [Li] for more details.

Now consider the surface $S = \psi_\alpha^{-1}(p)$ embedded in T_f^γ, and suppose that S has genus k. There exists a symplectic basis $\{x_i, y_i\}$, $1 \le i \le k$, of $H_1(S;\mathbb{Z}_2)$ such that x_k is homologous to the homology class $[\gamma]$ corresponding to γ in T_f^γ and y_k is homologous to the homology class of $\beta = S \cap \Sigma_{g,1} \subset T_f^\gamma$. But γ was required to have the spin structure that extends over a disk (so the spin structure on $T_{f,1}$ may be extended to a spin structure on T_f^γ.) Thus $q_\sigma(x_k) = q_\sigma([\gamma]) = 0$, and

$$\mathrm{Arf}(S, q_\sigma) = \sum_{i=1}^{k} q_\sigma(x_i)q_\sigma(y_i) = \sum_{i=1}^{k-1} q_\sigma(x_i)q_\sigma(y_i).$$

If we cut S open along a simple closed curve parallel to $\beta = S \cap \Sigma_{g,1}$ then the result deformation retracts to a surface S' with boundary $\partial S' = \beta \amalg -f(\beta)$ and with symplectic basis $\{x_i, y_i\}$, $1 \le i \le k-1$, of $H_1(S';\mathbb{Z}_2)$. If $q'_\sigma : H_1(S';\mathbb{Z}_2)/i_*(H_1(\partial S';\mathbb{Z}_2)) \to \mathbb{Z}_2$

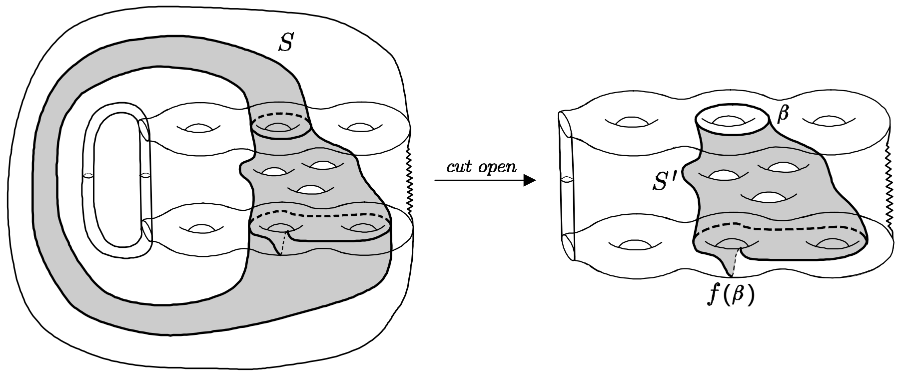


Figure 6.2: S cut open along β

is the induced $\mathbb{Z}_2$-quadratic form, then

$$\sum_{i=1}^{k-1} q_\sigma(x_i)q_\sigma(y_i) = \mathrm{Arf}(S, q_\sigma) = \mathrm{Arf}(S', q'_\sigma) = \sum_{i=1}^{k-1} q'_\sigma(x_i)q'_\sigma(y_i).$$

According to Johnson in [J1], the quadratic form q (in the statement of Theorem 6.2) corresponds to a Heegaard embedding of $\Sigma_{g,1}$ into S^3. Thus we get an induced embedding of $\Sigma_{g,1}\times[0,1]$ and then S' into S^3, and the quadratic form q'_σ is precisely the same as the mod 2 Seifert self-linking form. Thus we see that to calculate $\mathrm{Arf}(S, q_\sigma)$, we really only need to calculate the Arf invariant of the link $\{\beta, f(\beta)\}$ with Seifert surface S'.

Since there is an isomorphism $H^1(T_f^\gamma;\mathbb{Z}) \cong H^1(\Sigma_{g,1};\mathbb{Z})$, $\alpha \in H^1(T_f^\gamma;\mathbb{Z})$ has a corresponding class in $H^1(\Sigma_{g,1};\mathbb{Z})$ which we will also call α. The homology class of $\beta = S\cap\Sigma_{g,1}$ in $H_1(T_f^\gamma)$ also has a corresponding class in $H_1(\Sigma_{g,1})$ which we denote by $[\beta]$. Since the homology class of S is Poincaré dual to $\alpha \in H^1(T_f^\gamma;\mathbb{Z})$, $[\beta] \in H_1(\Sigma_{g,1})$ must be Poincaré dual to $\alpha \in H^1(\Sigma_{g,1};\mathbb{Z})$.

Proof of Theorem 6.2. We have a fixed spin structure on $\Sigma_{g,1}$. Let q be the associated

$\mathbb{Z}_2$-quadratic form. Recall from Chapter 3 that the hypothesis $\mathrm{Arf}(\Sigma_{g,1}, q) = 0$ was necessary for the Birman-Craggs homomorphism $\rho_q : \mathcal{J}(2) \to \mathbb{Z}_2$ to be defined. We have already seen that the spin structure on $\Sigma_{g,1}$ induces a spin structure σ on T_f^γ which in turn induces a spin structure $\sigma|_S$ on the surface S defined above. To prove the theorem, we need to find a primitive cohomology class $\alpha \in H^1(T_f^\gamma; \mathbb{Z})$ such that $\omega_\alpha(f) = \rho_q(f)$. To accomplish this we need to find a surface S such that $\mathrm{Arf}(S, q_\sigma) = \mathrm{Arf}(S', q'_\sigma) = \rho_q(f)$. To do so, we will construct a simple closed curve β on $\Sigma_{g,1}$ and calculate the Arf invariant $\mathrm{Arf}(\beta, f(\beta))$ with Seifert surface S'.

Recall that for genus $g = 2$ surfaces, the Torelli group $\mathcal{J}(2)$ is generated by the collection of all Dehn twists about bounding simple closed curves, and for genus $g \geq 3$, $\mathcal{J}(2)$ is generated by the collection of all Dehn twists about genus 1 cobounding pairs of simple closed curves, i.e. pairs of non-bounding, disjoint, homologous simple closed curves that together bound a genus 1 subsurface. Thus it is sufficient to prove the claim for such elements of $\mathcal{J}(2)$.

First assume that $g = 2$ and C is a genus 1 bounding simple closed curve on $\Sigma_{2,1}$. Let f be a Dehn twist about C. Then C splits $\Sigma_{2,1}$ into two genus 1 surfaces Σ_a and Σ_b. Let $\{x_a, y_a\}$ and $\{x_b, y_b\}$ be symplectic bases of $H_1(\Sigma_a)/i_*(H_1(\partial\Sigma_a))$ and $H_1(\Sigma_b)/i_*(H_1(\partial\Sigma_b))$, respectively. Then we have two cases:

(i) $\rho_q(f) = \mathrm{Arf}(\Sigma_a, q|_{\Sigma_a}) = \mathrm{Arf}(\Sigma_b, q|_{\Sigma_b}) = 1$

$\iff$ $q(x_a) = q(y_a) = q(x_b) = q(y_b) = 1$, or

(ii) $\rho_q(f) = \mathrm{Arf}(\Sigma_a, q|_{\Sigma_a}) = \mathrm{Arf}(\Sigma_b, q|_{\Sigma_b}) = 0$

$\iff$ at least one of $\{q(x_a), q(y_a)\}$ is 0 and at least one of $\{q(x_b), q(y_b)\}$ is 0.

Without loss of generality, let us assume in case (ii) that $q(x_a) = q(x_b) = 0$. In either case let β be a simple closed curve on $\Sigma_{2,1} \hookrightarrow T_f^\gamma$ which intersects C exactly twice and such that $[\beta] \in H_1(\Sigma_{2,1})$ is homologous to $x_a + x_b$. Then we also have the simple closed curve $f(\beta)$ on $f(\Sigma_{2,1}) \hookrightarrow T_f^\gamma$. Near C the picture will always be as in Figure 6.3, and we choose S' to be this particular surface pictured in Figure 6.3 with boundary $\partial S' = \beta \amalg -f(\beta)$. This surface S' has spin structure $\sigma|_{S'}$ and a

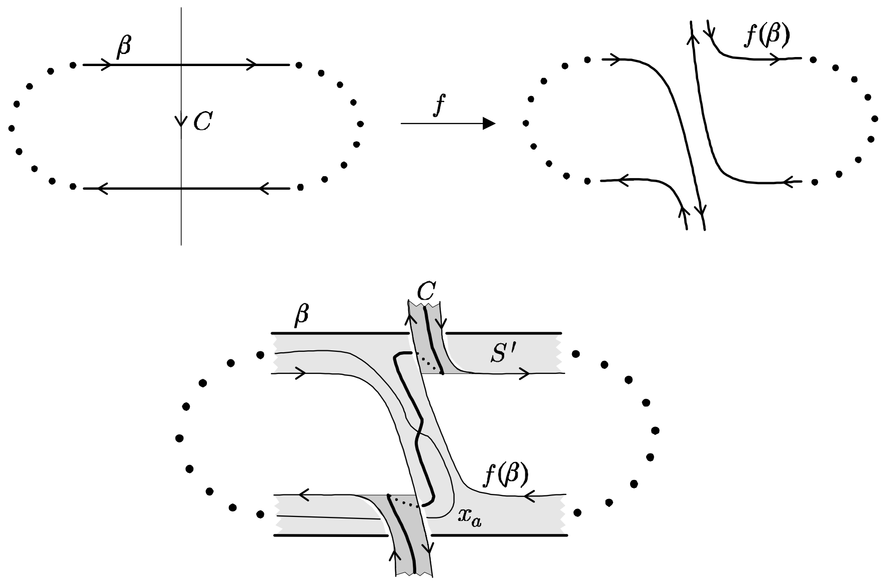


Figure 6.3: Surface S' in T_f^γ with boundary $\beta \amalg -f(\beta)$ (for $g = 2$)

corresponding quadratic form

$$q'_\sigma : H_1(S'; \mathbb{Z}_2)/i_*(H_1(\partial S'; \mathbb{Z}_2)) \to \mathbb{Z}_2$$

given by the mod 2 self-linking form. Notice that $\{x_a, [C]\}$ is a symplectic basis for the quotient $H_1(S'; \mathbb{Z}_2)/i_*(H_1(\partial S'; \mathbb{Z}_2))$. Then we have

$$\omega_\alpha(f) = \mathrm{Arf}(S', q'_\sigma) \overset{def.}{\equiv} \mathrm{Arf}(\beta, f(\beta)) = q'_\sigma(x_a) q'_\sigma([C]).$$

Note that, while C is a product of commutators on $\Sigma_{2,1}$, it is not a product of commutators on S'. But it is easy to see from Figure 6.3 that $q'_\sigma([C]) = \mathrm{lk}\,(C, C^+) \equiv 1$ modulo 2. It is also clear that $q'_\sigma(x_a) = q(x_a)$. Thus

$$\omega_\alpha(f) = q'_\sigma(x_a)q'_\sigma([C]) = q(x_a) = \mathrm{Arf}(\Sigma_a, q|_{\Sigma_a}) = \rho_q(f).$$

Now assume that $g \geq 3$ and C_1 and C_2 are genus 1 cobounding pairs of simple closed curves on $\Sigma_{g,1}$. Let f be a composition of Dehn twists about C_1 and C_2. Then C_1 and C_2 cobound a genus 1 subsurface Σ'. Let $\{x, y\}$ be a symplectic basis of $H_1(\Sigma')/i_*(H_1(\partial\Sigma'))$. There are two cases:

(1) $q(C_1) = q(C_2) = 1$ and

(2) $q(C_1) = q(C_2) = 0$.

For case (1), we simply let β be a simple closed curve on $\Sigma_{g,1} \hookrightarrow T_f^\gamma$ which does not intersect C_1 or C_2. Then f will not affect β, and we can choose S' to be a straight cylinder between β and $f(\beta)$ so that $H_1(S'; \mathbb{Z}_2)/i_*(H_1(\partial S'; \mathbb{Z}_2))$ is trivial. Thus $\omega_\alpha(f) = \mathrm{Arf}(S', q'_\sigma) = 0$. We also know from the end of Chapter 3 that in this case $\rho_q(f) = 0$.

For case (2), we have two subcases:

(i) $\rho_q(f) = \mathrm{Arf}(\Sigma', q|_{\Sigma'}) = 1 \iff q(x) = q(y) = 1$, or

(ii) $\rho_q(f) = \mathrm{Arf}(\Sigma', q|_{\Sigma'}) = 0 \iff$ at least one of $\{q(x), q(y)\}$ is 0.

Again without loss of generality, let us assume in case (ii) that $q(x) = 0$. In both cases let β be a simple closed curve on $\Sigma_{g,1} \hookrightarrow T_f^\gamma$ which intersects each of C_1

and C_2 exactly once and such that $[\beta] \in H_1(\Sigma_{g,1})$ is homologous to $x + x'$, where x' is any nontrivial homology class in $H_1(\Sigma_{g,1} - \Sigma')$. Then we also have the simple closed curve $f(\beta)$ on $f(\Sigma_{g,1}) \hookrightarrow T_f^\gamma$. Near C_1 and C_2 the picture will always be as in Figure 6.4, and we choose S' to be this particular surface pictured in Figure 6.4 with boundary $\partial S' = \beta \amalg -f(\beta)$. Again this surface S' has spin structure $\sigma|_{S'}$ and a

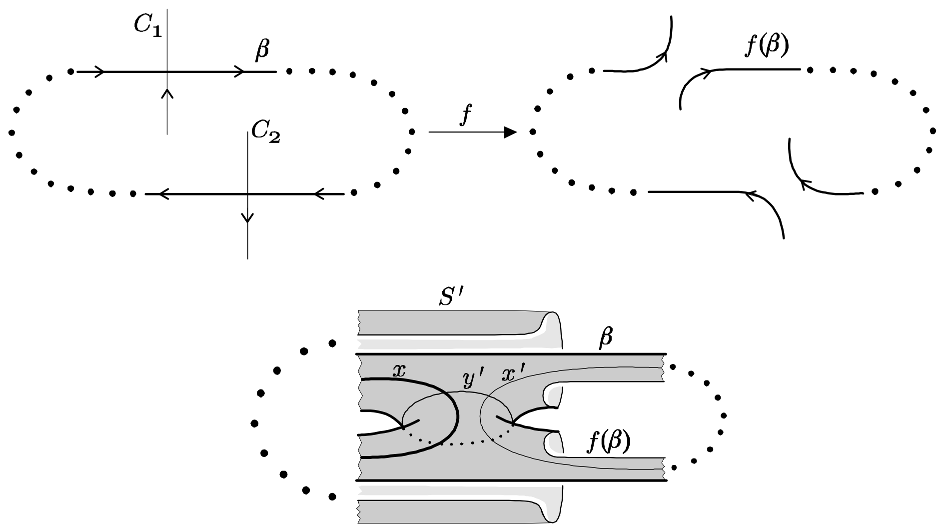


Figure 6.4: Surface S' in T_f^γ with boundary $\beta \amalg -f\,(\beta)$ (for $g \geq 3$)

corresponding quadratic form $q'_\sigma : H_1(S';\mathbb{Z}_2)/i_*(H_1(\partial S';\mathbb{Z}_2)) \to \mathbb{Z}_2$ given by the mod 2 self-linking form. Let y' be any homology class such that $\{x, y'\}$ is a symplectic basis for $H_1(S';\mathbb{Z}_2)/i_*(H_1(\partial S';\mathbb{Z}_2))$. Then we have

$$\omega_\alpha(f) = \mathrm{Arf}(S', q'_\sigma) \overset{def.}{\equiv} \mathrm{Arf}(\beta, f(\beta)) = q'_\sigma(x)q'_\sigma(y').$$

Notice that $\{x, y'\}$ is also a basis for $H_1(\Sigma';\mathbb{Z}_2)/i_*(H_1(\partial\Sigma';\mathbb{Z}_2))$ and that $q'_\sigma(x) = q(x)$

and $q'_\sigma(y') = q(y')$. Thus we see that

$$\omega_\alpha(f) = q'_\sigma(x) q'_\sigma(y') = q(x)q(y') = \mathrm{Arf}(\Sigma', q|_{\Sigma'}) = \rho_q(f).$$

This completes the proof of Theorem 6.2. □

As a result of this theorem and Theorem 5.14, we see that η_2 contains the necessary information for determining both the Johnson homomorphism τ_2 and the Birman-Craggs homomorphisms $\left\{\rho_q\right\}$. Recall from Chapter 4 that the abelianization $H_1(\mathcal{J}(2);\mathbb{Z}) \cong \mathcal{J}(2)/\left[\mathcal{J}(2), \mathcal{J}(2)\right]$ of the Torelli group is completely determined by $\left\{\tau_2, \rho_q\right\}$ since the commutator subgroup of the Torelli group is given by the kernels of $\left\{\tau_2, \rho_q\right\}$. Namely, we have $[\mathcal{J}(2), \mathcal{J}(2)] = \mathcal{C} \cap \mathcal{J}(3)$, where $\mathcal{C} = \bigcap_q \ker \rho_q$. Suppose we take a mapping class $f \in \ker \eta_2$. Certainly it is true that $f \in \mathcal{C} \cap \mathcal{J}(3)$ since τ_2 and $\left\{\rho_q\right\}$ factor through η_2. Of course it would be nice to know if the converse is also true.

Problem 6.3 *What is* $\ker \eta_2$*? Is it true that* $\ker \eta_2 = \mathcal{C} \cap \mathcal{J}(3) = [\mathcal{J}(2), \mathcal{J}(2)]$*?*

6.3 Analysis of η_k

In this section we shift our focus to the homomorphism $\eta_k : \mathcal{J}(k) \to \Omega_3^{spin}(F/F_k)$ for arbitrary values of k. We already know that $\ker \eta_k \subset \mathcal{J}(2k-1) = \ker \sigma_k$ since the original bordism homomorphism $\sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k)$ factors through $\Omega_3^{spin}(F/F_k)$. However, the additional structure on the bordism given by the spin structures may potentially refine the kernel of η_k.

Problem 6.4 *What is the kernel of* η_k*?*

Problem 6.5 *Does η_k give a faithful representation of the abelianization of $\mathcal{J}(k)$? In other words, is* $\operatorname{Im}\eta_k \cong \mathcal{J}(k)/\left[\mathcal{J}(k),\mathcal{J}(k)\right]$*?*

A sufficient condition for $f \in \ker\eta_k$ is given in the following theorem, but it is most likely not necessary. Consider the entire collection $\{\omega_\alpha\}$ of the homomorphisms $\omega_\alpha : \mathcal{J}(2) \to \Omega_3^{spin}(S^1)$ defined in Section 6.2, and let

$$\mathcal{B} = \bigcap_\alpha \ker\omega_\alpha$$

be the common kernel of all ω_α for all $\alpha \in H^1(T_f^\gamma;\mathbb{Z})$.

Theorem 6.6 *If $f \in \mathcal{B} \cap \mathcal{J}(2k+1)$, then $f \in \ker\eta_k$.*

Note that the hypothesis requires $f \in \mathcal{J}(2k+1)$, not just $f \in \mathcal{J}(2k-1)$. The purpose of this will be revealed in the proof of the theorem, but it is probably not necessary. However, since the kernel of σ_k is $\mathcal{J}(2k-1)$, it is certainly necessary that $f \in \mathcal{J}(2k-1)$.

Before we give the proof of this theorem, let us first set up some necessary notation. For a more complete discussion, we refer the reader to Whitehead's book [Wh]. We will be using the Atiyah-Hirzebruch spectral sequence. In particular, let

$$J_{p,q}^m = \operatorname{Image}\left(\left(i_{p,q}\right)_* : \tilde{\Omega}_{p+q}^{spin}\left(\left(\frac{F}{F_m}\right)^{(p)} \right) \to \tilde{\Omega}_{p+q}^{spin}\left(\frac{F}{F_m}\right) \right). \qquad (\star)$$

Here $(F/F_m)^{(p)}$ denotes the p-skeleton of $K(F/F_m,1)$, $\left(i_{p,q}\right)_*$ is induced by the inclusion map $(F/F_m)^{(p)} \hookrightarrow K(F/F_m,1)$, and $\tilde{\Omega}_n^{spin}(F/F_m)$ denotes the *reduced* spin bordism group defined by

$$\Omega_n^{spin}\left(\frac{F}{F_m}\right) \cong \Omega_n^{spin} \oplus \tilde{\Omega}_n^{spin}\left(\frac{F}{F_m}\right).$$

Note that if $(M, \phi, \sigma) \in J_{p,q}^m$ then for $l \leq m$ the triple $(M, \pi_{m,l} \circ \phi, \sigma)$ is in $J_{p,q}^l$, where $\pi_{m,l} : K(F/F_m, 1) \to K(F/F_l, 1)$ is the projection map. Let $E_{p,q}^2 \cong \tilde{H}_p(F/F_m; \Omega_q^{spin})$, and the boundary operator is $d_{p,q}^2 : E_{p,q}^2 \to E_{p-2,q+1}^2$. The groups $E_{p,q}^2$ may be thought of as the building blocks for $\tilde{\Omega}_n^{spin}(F/F_m)$ with $p + q = n$. In actuality, the building blocks are the groups $E_{p,q}^\infty = \lim E_{p,q}^r$, where for $r \geq 3$

$$E_{p,q}^r = \frac{\ker d_{p,q}^{r-1}}{\operatorname{Im} d_{p+r-1,q-r+2}^{r-1}} \text{ and } d_{p,q}^{r-1} : E_{p,q}^{r-1} \to E_{p-r+1,q+r-2}^{r-1}.$$

We also have an isomorphism

$$E_{p,q}^\infty \cong J_{p,q}^m / J_{p-1,q+1}^m. \qquad (\star\star)$$

Since $\Omega_3^{spin} = 0$, we then have $\Omega_3^{spin}(F/F_m) \cong \tilde{\Omega}_3^{spin}(F/F_m)$ and

$$\Omega_3^{spin}(F/F_m) = J_{3,0}^m \supseteq J_{2,1}^m \supseteq J_{1,2}^m \supseteq J_{0,3}^m = 0.$$

Then one can show that the relevant $E_{p,q}^\infty$ are as follows.

$$\begin{aligned}
E_{3,0}^\infty &= E_{3,0}^3 = \ker d_{3,0}^2 \subset \tilde{H}_3(F/F_m) \cong H_3(F/F_m) \\
E_{2,1}^\infty &= E_{2,1}^3 = \operatorname{coker} d_{4,0}^2 \cong H_2(F/F_m; \Omega_1^{spin}) / \operatorname{Im} d_{4,0}^2 \\
E_{1,2}^\infty &= E_{1,2}^4 = \operatorname{coker} d_{4,0}^3 \cong H_1(F/F_m; \Omega_2^{spin}) / \operatorname{Im} d_{3,1}^2 \\
E_{0,3}^\infty &= 0
\end{aligned}$$

We can now begin our proof of Theorem 6.6.

Proof of Theorem 6.6. We assume that $f \in \mathcal{B} \cap \mathcal{J}(2k+1)$, and we want to show that $\left(T_f^\gamma, \phi_{f,k}^\gamma, \sigma\right) = 0$ in $\Omega_3^{spin}(F/F_k)$. Since $\Omega_3^{spin}(F/F_k) = J_{3,0}^k$, we may perturb any $(M, \phi, \sigma) \in \Omega_3^{spin}(F/F_k)$ to ensure that $\phi(M)$ is contained in the 3-skeleton $(F/F_k)^{(3)}$ of $K(F/F_k, 1)$. That is, by the definition of $J_{3,0}^k$ given in $(\star)$ we can choose ϕ' homotopic to ϕ so that $(M, \phi', \sigma) \in \Omega_3^{spin}((F/F_k)^{(3)})$ and $(i_{3,0})_* (M, \phi', \sigma) = (M, \phi, \sigma)$ in $\Omega_3^{spin}(F/F_k)$.

So we start with $\left(T_f^\gamma, \phi_{f,k+1}^\gamma, \sigma\right) \in J_{3,0}^{k+1}$. Since $f \in \mathcal{J}(2k+1)$, Theorem 5.3 says that the pair $\left(T_f^\gamma, \phi_{f,k+1}^\gamma\right) = 0$ in $\Omega_3(F/F_{k+1})$. Thus $(\phi_{f,k+1}^\gamma)_*([T_f^\gamma]) = 0$ in $H_3(F/F_{k+1}) \cong E_{3,0}^\infty$, and we therefore know from $(\star\star)$ that $\left(T_f^\gamma, \phi_{f,k+1}^\gamma, \sigma\right)$ must be in $J_{2,1}^{k+1}$. Thus by $(\star)$ there exists a triple $(M', \phi', \sigma') \in \Omega_3^{spin}((F/F_{k+1})^{(2)})$ such that $(i_{2,1})_* (M', \phi', \sigma') = \left(T_f^\gamma, \phi_{f,k+1}^\gamma, \sigma\right)$ in $\Omega_3^{spin}(F/F_{k+1})$ as indicated in the following diagram.

$$
\begin{array}{ccccc}
 & & \Omega_3^{spin}((F/F_{k+1})^{(2)}) & & \\
 & & \downarrow & & (M', \phi', \sigma') \\
\Omega_3^{spin}(F/F_{2k+1}) & \longrightarrow & \Omega_3^{spin}(F/F_{k+1}) & & \downarrow \\
(T_f^\gamma, \phi_{f,2k+1}^\gamma, \sigma) & \longmapsto & & & (T_f^\gamma, \phi_{f,k+1}^\gamma, \sigma)
\end{array}
$$

Lemma 6.7 *The homomorphism*

$$\frac{J_{2,1}^{k+1}}{J_{1,2}^{k+1}} \xrightarrow{(\pi_{k+1,k})_*} \frac{J_{2,1}^{k}}{J_{1,2}^{k}}$$

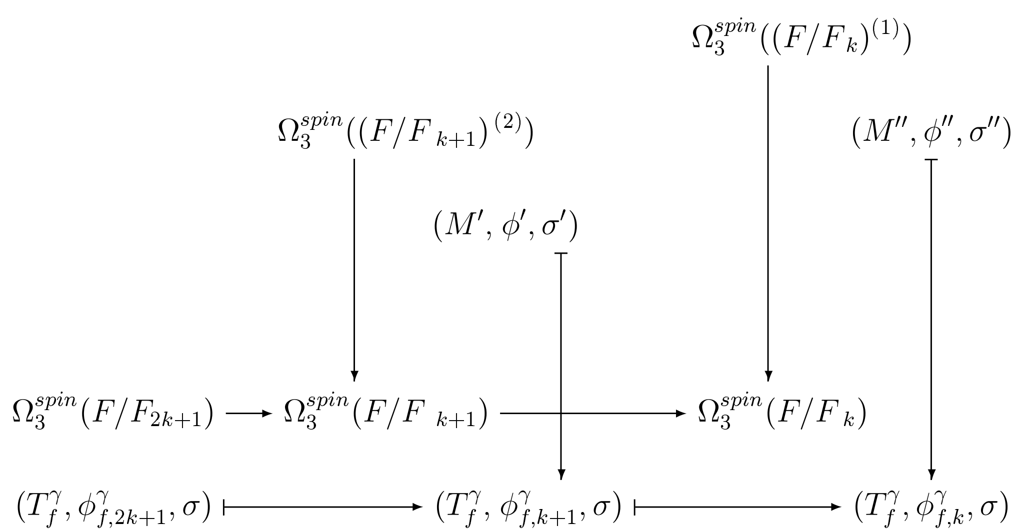
is the zero map.

Proof. By $(\star\star)$ we have $J_{2,1}^{k+1}/J_{1,2}^{k+1} \cong E_{2,1}^{\infty} \cong H_2(F/F_{k+1}; \Omega_1^{spin})/\operatorname{Im} d_{4,0}^2$. Similarly, we have $J_{2,1}^{k}/J_{1,2}^{k} \cong H_2(F/F_k; \Omega_1^{spin})/\operatorname{Im} d_{4,0}^2$. So this homomorphism is equivalent to

$$\frac{H_2(F/F_{k+1}; \Omega_1^{spin})}{\operatorname{Im} d_{4,0}^2} \to \frac{H_2(F/F_k; \Omega_1^{spin})}{\operatorname{Im} d_{4,0}^2}.$$

In the proof of Corollary 5.12 we showed that $H_2(F/F_{k+1}) \to H_2(F/F_k)$ is the zero map. Thus $H_2(F/F_{k+1}; \Omega_1^{spin}) \to H_2(F/F_k; \Omega_1^{spin})$ is also trivial, and the conclusion follows. □

Now let us consider the image of $\left(T_f^{\gamma}, \phi_{f,k+1}^{\gamma}, \sigma\right)$ in $\Omega_3^{spin}(F/F_k)$ under the homomorphism $(\pi_{k+1,k})_* : \Omega_3^{spin}(F/F_{k+1}) \to \Omega_3^{spin}(F/F_k)$. This image is of course $\left(T_f^{\gamma}, \phi_{f,k}^{\gamma}, \sigma\right)$, and since $\left(T_f^{\gamma}, \phi_{f,k+1}^{\gamma}, \sigma\right) \in J_{2,1}^{k+1}$ Lemma 6.7 tells us that we must have $\left(T_f^{\gamma}, \phi_{f,k}^{\gamma}, \sigma\right) \in J_{1,2}^{k}$. Then by $(\star)$ there exists a triple $(M'', \phi'', \sigma'') \in \Omega_3^{spin}((F/F_k)^{(1)})$ such that $(i_{1,2})_* (M'', \phi'', \sigma'') = \left(T_f^{\gamma}, \phi_{f,k}^{\gamma}, \sigma\right)$ in $\Omega_3^{spin}(F/F_k)$ as indicated in the following diagram.



Now we use the fact that $f \in \mathcal{B}$, the common kernel of all ω_α for all $\alpha \in H^1(T_f^\gamma;\mathbb{Z})$. Recall that $\omega_\alpha = (\pi_\alpha)_* \circ \eta_2 : \mathcal{J}(2) \to \Omega_3^{spin}(S^1)$ and $(\pi_\alpha)_* : \Omega_3^{spin}(F/F_2) \to \Omega_3^{spin}(S^1)$. Since the 1-skeleton $(F/F_k)^{(1)}$ is homotopy equivalent to the wedge of $2g$ circles, we have

$$
\begin{aligned}
\Omega_3^{spin}((F/F_k)^{(1)}) &\cong \Omega_3^{spin}(S^1 \vee \cdots \vee S^1) \\
&\cong \overset{2g}{\oplus}\Omega_3^{spin}(S^1)
\end{aligned}
$$

and the following commutative diagram.

$$
\begin{array}{ccccc}
 & & \overset{2g}{\oplus}\Omega_3^{spin}\left(S^1\right) & \longrightarrow & \Omega_3^{spin}\left(S^1\right) \\
 & & \big\downarrow (i_{2,1})_* & & \big\uparrow (\pi_\alpha)_* \\
\mathcal{J}(k) & \xrightarrow{\eta_k} & \Omega_3^{spin}(F/F_k) & \xrightarrow{(\pi_{k,2})_*} & \Omega_3^{spin}(F/F_2)
\end{array}
$$

There is a basis of $H^1(T_f^\gamma;\mathbb{Z})$ such that for each basis element α_i the range of the homomorphism ω_{α_i} corresponds to a summand of $\Omega_3^{spin}((F/F_k)^{(1)}) \cong \overset{2g}{\oplus}\Omega_3^{spin}(S^1)$. Since $f \in \mathcal{B}$, $(M'',\phi'',\sigma'') \in \Omega_3^{spin}((F/F_k)^{(1)})$ is trivial in each summand of $\overset{2g}{\oplus}\Omega_3^{spin}(S^1)$, and thus it is trivial in $\Omega_3^{spin}((F/F_k)^{(1)})$. Therefore $0 = (i_{1,2})_*(M'',\phi'',\sigma'') = \left(T_f^\gamma, \phi_{f,k}^\gamma, \sigma\right)$ in $\Omega_3^{spin}(F/F_k)$. $\square$

Bibliography

[BC] J. Birman and R. Craggs, *The μ-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold*, Trans. of the AMS 237 (1978) 283-309.

[CGO] T. Cochran, A. Gerges, and K. Orr, *Dehn surgery equivalence relations on 3-manifolds*, Math. Proc. Camb. Phil. Soc. 131 (2001) 97-127.

[Dw] W. Dwyer, *Homology, Massey products, and maps between groups*, J. Pure and Applied Algebra 6 (1975) 177-190.

[Fe] R. Fenn, *Techniques of Geometric Topology*, London Mathematical Society Lecture Notes Series 57, Cambridge University Press (1983).

[IO] K. Igusa and K. Orr, *Links, pictures and the homology of nilpotent groups*, Topology 40, Issue 6 (2001) 1125-1166.

[J1] D. Johnson, *Quadratic forms and the Birman-Craggs homomorphisms*, Trans. of the AMS 261, no. 1 (1980) 235-254.

[J2] _____, *An abelian quotient of the mapping class group $\mathcal{T}_g$*, Math. Annalen 249 (1980) 225-242.

[J3] _____, *A survey of the Torelli group*, Contemporary Math. 20 (1983) 163-179.

[J4] _____, *The structure of the Torelli group I: A finite set of generators for $\mathcal{T}_g$*, Annals of Math. 118 (1983) 423-442.

[J5] _____, *The structure of the Torelli group II: A characterization of the group generated by twists on bounding curves*, Topology 24, Issue 2 (1985) 113-126.

[J6] _____, *The structure of the Torelli group III: The abelianization of $\mathcal{T}_g$*, Topology 24, Issue 2 (1985) 127-144.

[Ki] T. Kitano, *Johnson's homomorphisms of subgroups of the mapping class group, the Magnus expansion and Massey higher products of mapping tori*, Topology and its Applic. 69 (1996) 165-172.

[Kr] D. Kraines, *Massey Higher Products*, Trans. of the AMS 124 (1966) 431-449.

[Li] W.B.R. Lickorish, *An Introduction to Knot Theory*, Springer-Verlag (1997).

[Me] G. Mess, *The Torelli groups for genus 2 and 3 surfaces*, Topology 31 (1992) 775-790.

[M1] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*, Duke Math. Journal 70 (1993) 699-726.

[M2] _____, *The extension of Johnson's homomorphism from the Torelli group to the mapping class group*, Inventiones Math. 111 (1993) 197-224.

[Or] K. Orr, *Link concordance invariants and Massey products*, Topology 30, Issue 4 (1991) 699-710.

[St] J. Stallings, *Homology and central series of groups*, J. Algebra 2 (1965) 170-181.

[Su] D. Sullivan, *On the intersection ring of compact 3-manifolds*, Topology 14 (1975) 275-277.

[Tu] V. Turaev, *Nilpotent homotopy types of closed 3-manifolds*, Proceedings of Topology Conference in Leningrad, Lecture Notes in Mathematics 1060 (1982), Springer-Verlag, 355-366.

[Wh] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag (1978).