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Understand principles and properties of the set of complex numbers and its subsets

The set of complex numbers:

- The set of all numbers of the form $a+bi$ such that $a,b \in \mathbf{R}$.
  - i represents the number $\sqrt{-1}$.
- The complex numbers have many subsets, the ones we are most concerned with are the:
  - Whole Numbers or Natural Numbers (**N**)
    - {0, 1, 2, 3, 4…}
  - Integers (**Z**)
    - {…-2, -1, 0, 1, 2, 3…}
  - Rational Numbers (**Q**)
    - $\left\{ a/b : a,b \in \mathbf{Z} \right\}$
  - Irrational Numbers
    - Numbers that cannot be represented in the form $\left\{ a/b : a,b \in \mathbf{Z} \right\}$
  - Real Numbers (**R**)
    - The set of all Rational and Irrational numbers

Principles of Number Theory:

- Prime Numbers:
  - Natural Numbers divisible only by one and itself
    - Ex. Factors of 13 {1, 13}
- Composite numbers:
  - Natural Numbers divisible by integers other than one and itself
    - Ex. Factors of 14 {1, 2, 7, 14}
- Divisibility:
  - An integer *a* is considered divisible by *b* if $a/b = c : c \in \mathbf{Z}$
  - We often write $b|a$ if $a/b = c : c \in \mathbf{Z}$
    - We pronounce $b|a$ as b divides a
  - Common divisors or factors are factors that two integers share in common
    - NOTE: Factors of integers are always positive
    - If $b|a$ and $b|c$ then *b* is a common factor among *a* and *c*
    - The greatest common divisor is the largest common factor shared among two numbers
      - This is denoted as gcd(*a*,*b*) = *d*
        - Ex. gcd(15, 20) = 5
        - 5 is the largest factor in common
      - The gcd of two numbers is also the greatest number of prime factors shared in common among the numbers
    - If gcd(*a*,*b*) = 1 the numbers *a* and *b* are considered relatively prime
      - If two numbers are prime their gcd = 1

Number Concepts:

- Fractions
  - Proper fractions $\left(a/b : b > a \text{ and } a, b \in \mathbf{Z}\right)$
  - Improper fractions $\left(a/b : b < a \text{ and } a, b \in \mathbf{Z}\right)$
  - Mixed Numbers $\left(c\,a/b : b > a \text{ and } a, b, c \in \mathbf{Z}\right)$
    - Converting from mixed number to improper fraction for solving algebraic equations:
      - $c\,a/b = \frac{cb+a}{b}$
  - Convert from fraction to decimal
    - Long divide the numerator by the denominator and add place holders while dividing until remainder is zero
      - Ex. $\frac{3}{4} = 4\overline{)3.00}$ with quotient .75

        -2.8
        .20
        -.20
        0
    - Note if remainder is not going to zero look for a pattern in the decimal and make it repeating.
- Repeating decimals and Percents
  - Repeating decimals are rational numbers because they can be represented in fraction form
  - Converting to fractions
    - Let *a* equal the repeating decimal
      - *a* = .32323232…
    - Multiply *a* by some scalar *n* to make the decimal of *na* look the same as *a*
      - 100*a* = 32.323232…
    - If we subtract *a* from *na* we get $(n-1)a$ equal to some terminating decimal
      - 99*a* = 32
    - Now we solve for *a* and that is our decimal in fraction notation
      - $a = \frac{32}{99}$
  - Percents when taken in context are converted to decimals
    - Remove the percent symbol and move the decimal to the left two digits
- Exponents
  - Rules to know
    - $x^n x^m = x^{n+m}$
    - $\left(x^n\right)^m = x^{nm}$
    - $\frac{x^n}{x^m} = x^{n-m}$
    - $x^0 = 1$

        - $\sqrt[n]{x} = x^{\frac{1}{n}}$
    - When two numbers raised to a variable have the same base and set equal to each other their exponents are equal
        - If $a^x = a^y$ then $x = y$

Need For Given Number Systems:

- Representation started with counting numbers which is all integers greater than 0. There is no identity element in the counting numbers under the operation of addition. Also 0 acts a place holder. Therefore we created natural numbers
- If 0 is considered the identity there is no additive inverse in the natural numbers. If there was some number *a*>0 what do we add to *a* in order for us to get zero. This was the birth of negative numbers so that *a* + (-*a*) = 0. These are the integers.
- Integers are not closed under the operation of division though. Consider a/b where *b* does not divide *a*. This is not an integer so now we have a new set called the rational numbers
- If we have a square with area of 2 what is the length of the side. We know area of a square is derived by the side squared, but what is $\sqrt{2}$? $\sqrt{2}$ cannot be represented in the form a/b where *a* and *b* are integers. Therefore we have the set of Irrational numbers.
- We could find square roots of all numbers greater than 0 but what about the negative numbers? $\sqrt{-1}$ was an important value because it helps solve cubic equations. $\sqrt{-1}$ is given a value which we refer to as *i*. Complex numbers came from this when we were trying to find roots of quadratic equations.

Complex Numbers: (http://www.clarku.edu/~djoyce/complex/)

- When working in rectangular coordinates we represent complex numbers in the form $a+bi$ such that $a, b \in \mathbf{R}$
- Under the complex numbers there are *n* roots of a function of degree *n*. This is more commonly known as the fundamental theorem of algebra.
- In rectangular coordinates complex roots will always come in conjugate pairs. Thus there always must be an even number of complex roots for a function
    - Conjugate pairs are $a+bi$ and $a-bi$
- Graphically we can represent complex numbers in the (x, yi) axes.
    - Plot coordinates and draw a line connecting the point to the origin
    - This is a vector notation of the value $a+bi$
- To add complex numbers you must combine similar terms as you would in an algebra problem with variables.
    - $(a+bi)+(c+di)=(a+c)+(b+d)i$
- Subtracting is very similar
    - $(a+bi)-(c+di)=(a-c)+(b-d)i$
- To multiply complex numbers you must distribute as would with binomials.

    - Recall that $i = \sqrt{-1}$ so $i^2 = -1$
    - $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$ because $bdi^2 = -bd$
- Addition of complex numbers represented graphically is vector addition represented graphically.
    - Take the tail of one of the vectors add translate it to the head of the other vector
        - Addition is commutative so the same result will come from placing them in opposite orders
    - To subtract complex numbers graphically place the tails of the two vectors together and draw the vector that connects them at the head
        - Direction is determined by which vector you subtract first. Place the tail of the new vector at the vector you subtract from. Then translate that vector to the origin to get the new coordinates
        - As you can see subtraction is not commutative so order is important
    - Multiplication by scalar dilates the vector
- When working in polar coordinates we represent complex numbers in the form $r(\cos q + i \sin q)$ or re^{iq} where $r = \sqrt{a^2 + b^2}$ and $q = \arctan b/a$
    - This is better viewed graphically where r is the distance from the origin or the radius of the vector, and q is the angle between the vector and the positive x-axis.
- When multiplying to complex numbers in polar form we must remember to use rules of exponents
    - $(r_1 e^{iq})(r_2 e^{ij}) = r_1 r_2 e^{i(q+j)}$
    - Notice we add the angles and multiply the radii

Group Theory: (http://members.tripod.com/~dogschool/groups.html)

- Groups are set of elements under a certain operation. They must satisfy the following axioms (http://members.tripod.com/~dogschool/groups.html)
    - The set is closed under the operation
        - If $a, b \in S$ then $a \bullet b \in S$
    - The set is associative under the operation
        - $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
    - The set contains an identity element
        - $a \bullet e = e \bullet a = a$
    - Every element has its inverse within the set
        - $a \bullet a^{-1} = e$ and $a^{-1} \bullet a = e$ where e is the identity element
- Groups do not need to be commutative. If a group is commutative it is called an abelian group.
    - This means $xy = yx$
- Important facts about groups. Let {G, *} be a group and $x, y \in G$:
    - $(xy)^n \neq x^n y^n$ unless G is abelian
    - $(xy)^{-1} \neq x^{-1} y^{-1}$ unless G is abelian

- $(xy)^{-1} = y^{-1}x^{-1}$ is always true

Field Theory:

- Fields are similar to groups but they work under addition and multiplication. Fields must satisfy the following axioms
  - Closed under both operations
  - Commutative under both operations
  - Associative under both operation
  - There must exist an additive identity element
  - There must exist a multiplicative identity element
  - The Distributive Property must apply
    - $c(a+b) = ca + cb$
  - There are no zero divisors
    - If $xy = 0$ then x or $y = 0$
  - Every non-zero element is a unit
    - Every element must have an inverse in the set
      - $\forall x \exists y \mid xy = 1$

1. Are the set of Natural Numbers contained within the set of Complex Numbers? Why or why not? (Explain using the rectangular coordinate definition of complex numbers)

The set of natural numbers are contained within the complex numbers. All complex numbers are written in the form $a + bi$ such that $a, b \in \mathbf{R}$. Now consider the case where b is always zero. This leaves us with $a + 0i$ and we know zero times anything is zero. We also know zero is the additive identity so now we are left with a. We are left with all real numbers a. We also know the real numbers are the union of the irrationals and rationals. Lets restrict it to the rationals. Rational numbers are of the form $\{a/b : a, b \in \mathbf{Z}\}$, so lets restrict this to the case where b = 1. So now we have all integers. From here we restrict a to be greater than or equal to zero leaving us with the set of natural numbers.

2. Let p be some prime number and $k \in \mathbf{Z}$. What do we know about the value of d if gcd(p,k) = d and why.

The value of d can only be 1 or p. Since p is a prime number the only two factors it has are 1 and p. In fact the only time that $d = p$ is when k is a multiple of p.

3. Solve for x: $3^{x+1} = 27^{2x}$.

Recall that if numbers with the same base are raised to a power and set equal their exponents are equal. 3 is prime so that cannot be factored, but 27 = 3^3. So we can rewrite the equation like this.

$3^{x+1} = (3^3)^{2x}$	Using rules of exponents we know $(3^3)^{2x} = 3^{6x}$ so
$3^{x+1} = 3^{6x}$	Now we have numbers with the same base on each side so
$x+1 = 6x$	Get variables on one side and constants on the other side
$1 = 5x$	Divide both sides by 5 to get
$x = 1/5$	

4. Convert $4\,5/8$ to an improper fraction and then into decimal notation

$$4\,5/8 = 4 + 5/8 = \frac{4\cdot 8}{8} + \frac{5}{8} = \frac{32+5}{8} = \frac{37}{8}$$

$$\frac{37}{8} = 37 \div 8 = 37\overline{)8.000000}\quad .216216 \qquad .\overline{216}$$

-7.4
.60
-.37
.230
-.222

5. Convert $.\overline{09}$ to a fraction.

Let n = .09090909…
Let $100n$ = 9.09090909…
Thus if we subtract the two equations we get

$100n - n$ = 9.090909…-.090909…

The repeating decimals cancel out so we get

$99n = 9$

If we solve for n we get

$n = 9/99$

Simplify the expression to get

$n = 1/11 = .\overline{09}$

6. Perform the following algebraically and check it graphically. $2(2+3i)+(-9+6i)$

Algebraically:

Distribute the scalar first $\quad 2(2+3i) = 4+6i$

Now add the numbers $\quad (4+6i)+(-9+6i) = (4-9)+(6+6)i = -5+12i$

Graphically: see attached

7. Determine whether $\{S, @\}$ is a group based on the operation table and explain why.

@	a	b	c
a	a	b	c
b	b	a	b
c	c	b	a

First we must examine if the set is closed, which it is.
a is the identity element.
Every element is its own inverse
The operation is commutative even though it does not need to be
But the operation is not associative. $(c @ b) @ b \neq c @ (b @ b)$
Therefore this is not a group.

8. Is $S = \{ a + bi \mid a, b \in \mathbf{Q} \}$ a field. Why or why not?

First we must determine if the set is closed under addition and multiplication.
$(a+bi)+(c+di)=(a+c)+(b+d)i$ shows it is closed under addition and addition is always commutative.
$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$ shows it is closed under multiplication and multiplication is commutative in the rationals so it is commutative here.
The additive identity exists when a and b are zero
The multiplicative identity exists when a is 1 and b is 0
We saw the distributive property in action with scalars
Most important is to find the inverse of $a+bi$
We want $(a+bi)(c+di)=(ac-bd)+(ad+bc)i = 1+0i$
So we want $\begin{matrix} ac-bd=1 \\ ad+bc=0 \end{matrix}$ and we want to solve for c and d in terms of a and b.
After some linear substitutions we end with $c = \dfrac{a}{a^2+b^2}$ and $d = \dfrac{-b}{a^2+b^2}$
These are rational numbers as long as $a^2+b^2 \neq 0$, but that can only happen if a and b are both 0 and we know that the zero element has no inverse.
So S is a field.