Finite Mathematics Problem Set G Solutions

PSG

5.3 4. Raskins and Bobbins Ice Cream Shop makes three kinds of ice cream using skim milk, cream, vanilla, and cacao. Each gallon of Deluxe Vanilla uses 3 quarts of milk, 1 quart of cream, and 2 ounces of vanilla. Each gallon of Regular Vanilla uses 3.5 quarts of milk, 0.5 quarts of cream, and 1 ounce of vanilla. Each gallon of Deluxe Chocolate uses 3.25 quarts of milk, 0.75 quart of cream, and 2 ounces of cacao. How many gallons of each type of ice cream should be made in order to use up 100 gallons of milk, 25 gallons of cream, 5 pounds of vanilla, and 10 pounds of cacao?

Because we need practice of new ways, I will solve this using row reduction of matrices, as discussed in §2.2. I will let the gallons of deluxe vanilla be denoted as d, the gallons of regular vanilla be r, and the gallons of deluxe chocolate as c. We have one equation for each ingredient:

Milk: $3d + 3.5r + 3.25c = 400$ (this is all in quarts, so 4(100))

Cream: $d + 0.5r + 0.75c = 100$ (again, all in quarts)

Vanilla: $2d + r = 80$ (in ounces, so 16(5))

Cacao: $2c = 160$ (again in ounces)

We can encode this into an augmented matrix: $\left[\begin{array}{ccc|c} 3 & 3.5 & 3.25 & 400 \\ 1 & 0.5 & 0.75 & 100 \\ 2 & 1 & 0 & 80 \\ 0 & 0 & 2 & 160 \end{array}\right]$.

In order to eliminate decimals, and to simplify the last row, I will multiply the first two by 4 and divide the last by 2 to get: $\left[\begin{array}{ccc|c} 12 & 14 & 13 & 1600 \\ 4 & 2 & 3 & 400 \\ 2 & 1 & 0 & 80 \\ 0 & 0 & 1 & 80 \end{array}\right]$.

Next I will subtract 13 times the last row from the first, and 3 times the last row from the second: $\left[\begin{array}{ccc|c} 12 & 14 & 0 & 560 \\ 4 & 2 & 0 & 160 \\ 2 & 1 & 0 & 80 \\ 0 & 0 & 1 & 80 \end{array}\right]$.

Now I will notice that the second and third are multiples of each other, and subtract two times the third from the second to get all zeroes, and place that result in the last row, while moving the third to the first, producing: $\left[\begin{array}{ccc|c} 2 & 1 & 0 & 80 \\ 12 & 14 & 0 & 560 \\ 0 & 0 & 1 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

To make another zero in the first column, I will multiply the first row by 6 and subtract from the second to get: $\left[\begin{array}{ccc|c} 2 & 1 & 0 & 80 \\ 0 & 8 & 0 & 80 \\ 0 & 0 & 1 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

We're making good progress, next I will divide the second row by 8 to see: $\left[\begin{array}{ccc|c} 2 & 1 & 0 & 80 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

And then subtract the second from the first to get: $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 70 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Finally, divide the first row by 2 yielding: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 35 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

This means $d = 35$, $r = 10$, and $c = 80$. I should check this, but please notice that we did some things in sequences you might have found unfamiliar. Look back and see that I did that in order to avoid decimals or fractions. Either way is fine. Now, to checking:
Milk: $3(35) + 3.5(10) + 3.25(80) = 400$ (this is all in quarts, so $4(100)$)
Cream: $(35) + 0.5(10) + 0.75(80) = 100$
Vanilla: $2(35) + (10) = 80$
Cacao: $2(80) = 160$
All looks good.

5.3. 21. Robin makes bows and arrows using wood, string, and feathers. Each bow uses 5 feet of wood and 4 feet of string, while each arrow uses 3 feet of wood and 4 feathers. If Robin has 100 feet of wood and 32 feet of string, how many feathers does he need so that he can use up all the wood, string, and feathers making bows and arrows?

This problem is a little different because of the question about feathers. I'm going to start by ignoring feathers. I'm using b for bows and a for arrows.
Wood: $5b + 3a = 100$
String: $4b = 32$
I would like to use matrices for practice, but that seems silly here. I hope it is clear that $b = 8$ by dividing by 4. From that $5(8)+3a = 100$ i.e. $40+3a = 100$ so $3a = 60$ and $a = 20$. To make 20 arrows, Robin needs $4(20) = 80$ feathers. It's probably not bad to check, but pretty direct that $5(8) + 3(20) = 100$ and $4(8) = 32$.

5.3 28. A student is trying to decide how to allocate her study time among mathematics, English, biology, and economics. She decides to spend a total of 45 hours per week studying and to spend twice as much time on mathematics and biology combined as on English and economics combined. Also, she will spend twice as much time on economics as on English and the same amount of time on mathematics as biology. How much time will she spend on each subject per week?

I'm going to use m for mathematics, g for English, b for biology, and c for economics. The first half of the second sentence says all together they are 45 hours, so $m+g+b+c = 45$. The second half says $2(g + c) = m + b$ (be careful to read that; notice the way it's written that math. and bio. are bigger than the other two, so it needs to be this way). The third sentence also has two parts. The first part says $2g = c$. The second part simply says $m = b$.

We can move all the variables to the left side to get these equations:
$m + g + b + c = 45$
$m - 2g + b - 2c = 0$
$2g - c = 0$
$m - b = 0$

I'm still wanting to practice matrices, so that's what I'm going to do: $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 45 \\ 1 & -2 & 1 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{array}\right]$.

Next I will subtract the first row from the second and fourth rows: $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 45 \\ 0 & -3 & 0 & -3 & -45 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & -2 & -1 & -45 \end{array}\right]$.

Now to clean up the numbers, I'll multiply the second and third by (-1) and also switch them: $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 45 \\ 0 & 1 & 2 & 1 & 45 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 3 & 45 \end{array}\right]$.

And while I'm making nice, I'll divide the last by 3 (I could have done this in one step, but honestly I didn't see it, and there's no harm in more steps): $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 45 \\ 0 & 1 & 2 & 1 & 45 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 15 \end{array}\right]$.

And, since that worked out so nicely, I'm switching them back. Ok, that was silly, I admit. Probably it's best to just see it originally, but this is honest work: $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 45 \\ 0 & 1 & 0 & 1 & 15 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 45 \end{array}\right]$.

Now, I'm in a good place to subtract the second from the first, the fourth, and two times the second from the third. I hope you're fine with all that at once. Watch each row: $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 30 \\ 0 & 1 & 0 & 1 & 15 \\ 0 & 0 & 0 & -3 & -30 \\ 0 & 0 & 2 & 0 & 30 \end{array}\right]$.

This time I see more clearly, so I will divide the third by -3 and the fourth by 2, and switch them to get: $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 30 \\ 0 & 1 & 0 & 1 & 15 \\ 0 & 0 & 1 & 0 & 15 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$.

Finally I subtract the third from the first and the second from the fourth to see: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 15 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$.

This gives us 15 hours for mathematics, 5 hours for English, 15 hours for biology and 10 hours for economics. They do add to 45, and mathematics and biology are equal. Furthermore economics is twice English, and finally mathematics and biology (30 hours) is twice English and economics (15 hours), and it all checks.

6.1 15. Let matrices A, B, and C be defined by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$$

Decide which of the following operations are defined, and carry out those which are defined.

(a) $3A - B = 3\begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -9 \\ 3 & 12 & 18 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -2 & -9 \\ 2 & 8 & 15 \end{bmatrix}.$

(b) $CA = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -3 \\ 4 & 0 & -6 \end{bmatrix}$

(c) $(A - 2B)C$. While $A - 2B$ can be computed, much the same as the first question, it cannot be multiplied by C. The entries do not match up in the correct way. The length of the first matrix (3) is not the same as the height of the second matrix (2).

(d) ABC. None of these multiplications can be done. We cannot multiply AB and we cannot multiply BC for all the same reasons as the above.

6.1.21. Find three 2×2 matrices, A, B, and C (C not all zeroes) such that $AC = BC$ but $A \neq B$.

Here is a simple answer

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

See $AC = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = BC.$

If you think that we need to have zeroes, here is an example with no zeroes:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -1 & 4 \\ 2 & 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

See $AC = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} = BC.$