Finite Mathematics Problem Set H Solutions

6.2 16. (a) Express the following system of equations in matrix form:

$$2x_1 - 2x_2 = 2$$
$$3x_1 - 2x_3 = 4$$
$$2x_2 - x_3 = -2$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 2 \\ 3 & 0 & -2 & 4 \\ 0 & 2 & -1 & -2 \end{array}\right]$$

(b) Find the inverse of the coefficient matrix of (a).

To do this we place it aside an identity matrix and reduce:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 0 & 1 & 0 & 0 \\ 3 & 0 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

I'm going to write steps in words, not symbols. This time I won't be scared of fractions. I hope you're ok with that. To get an upper-left 1, first I will multiply the top row by $\frac{1}{2}$:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 3 & 0 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Now that we have that 1, I will multiply the top row by 3 and subtract it from the second row to make 0s in the first column:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 3 & -2 & -\frac{3}{2} & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

I like the last row a little better than the third, so I will switch them:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 3 & -2 & -\frac{3}{2} & 1 & 0 \end{array}\right]$$

Now I would like a 1 in the middle instead of that two, so I will again multiply by $\frac{1}{2}$. I know, more fractions. I'm sorry. They are rather unavoidable with inverses, just like with numbers. Here we go:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 3 & -2 & -\frac{3}{2} & 1 & 0 \end{array}\right]$$

For a not-too-bad step, we can add the middle row to the first to give one zero closer to the identity on the left side:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 3 & -2 & -\frac{3}{2} & 1 & 0 \end{array}\right]$$

These steps don't get any nicer, I guess. Best to just do and make it through. Next multiply the middle row by 3 and subtract from the bottom to make one more zero:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 1 & -\frac{3}{2} \end{array}\right]$$

I really want to multiply the bottom row by -2 to make a 1 and remove some fractions, but I see something before that. I see that all the third column is $-\frac{1}{2}$, so before I do anything else, I'm going to cancel some of those out. First I will subtract the bottom from the top:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 1 & -\frac{3}{2} \end{array}\right]$$

That was nice. Let's try that again, subtracting the bottom from the middle:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & \frac{3}{2} & -1 & 2 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 1 & -\frac{3}{2} \end{array}\right]$$

And now we complete by returning to the delayed step of multiplying the bottom row by -2:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & \frac{3}{2} & -1 & 2 \\ 0 & 0 & 1 & 3 & -2 & 3 \end{array}\right]$$

That's probably the most work we've done in this course. Probably you should check by multiplying in both directions. I know I am. And I found a mistake in doing so. Checking is important. So, the inverse is $\begin{bmatrix} 2 & -1 & 2 \\ \frac{3}{2} & -1 & 2 \\ 3 & -2 & 3 \end{bmatrix}$

(c) Use the inverse of the coefficient matrix determined in (b) to solve the system of equations.

That was an unreasonable amount of work for doing only this. Solving the original three equations is much easier using any other method. *However* if you wanted to re-solve repeatedly with different values for the constants on the right (for example if they were supplies for making something, as we have seen before) then using inverses would be helpful because you only need to do it once. We originally have $A\vec{x} = \vec{b}$. We multiply both sides on the left by A^{-1} to get $A^{-1}A\vec{x} = A^{-1}\vec{b}$. On the left side of the equation $A^{-1}A = I$ so $A^{-1}A\vec{x} = Ix = x$ which gives us that $x = A^{-1}\vec{b}$. So, we compute using our inverse and list of constants: $\begin{bmatrix} 2 & -1 & 2 \\ \frac{3}{2} & -1 & 2 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -8 \end{bmatrix}$. Again we have done much work, we should check this. $2(-4) - 2(-5) = 2\checkmark$, $3(-4) - 2(-8) = 4\checkmark$, and $2(-5) - (-8) = -2\checkmark$. That was a long problem. I'm guessing we're glad it's over. Make sure you understand it all.

There's one thing that was particularly curious in (b). We spent a lot of time with fractions, and most of them went away in the end. I will give two extra points on XM567 to anyone who sends me full details by email no later than 7 May of a step-by-step process that uses far fewer fractions. The best possible would be no fractions until the very last step. That would surprise me, but I do believe we can do better than I did.

6.2 17. Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. Find A^{-1} and then the inverse of A^{-1}, denoted by $(A^{-1})^{-1}$ using the methods of class.

I'm tired of using row reduction, although I want you to be able to do it for even 2×2 matrices. I will complete this problem set by using the formula instead. $A^{-1} = \frac{1}{5(1)-2(2)} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$. Now to find the inverse of A^{-1}, $(A^{-1})^{-1} = \frac{1}{1(5)-(-2)(-2)} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, the matrix we started with. This is worth noticing, always $(A^{-1})^{-1} = A$.

6.2 18. Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. Find $A^{-1}, B^{-1}, (AB)^{-1}$, and $(BA)^{-1}$.

Lots of inverses, good practice. These problems are intentionally generous and we did A^{-1} in the one above. (We're making up for the first problem being long on the rest of the problem set.) $B^{-1} = \frac{1}{1(5)-2(3)} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (notice that the fraction in front is equal to -1 this time.

Multiplying we get $AB = \begin{bmatrix} 9 & 25 \\ 4 & 11 \end{bmatrix}$ and $BA = \begin{bmatrix} 11 & 5 \\ 20 & 9 \end{bmatrix}$ (notice a nice example of how non-commutative matrix multiplication is - we get different answers that aren't really related when we multiply in different orders).

Finally $(AB)^{-1} = \frac{1}{9(11)-4(25)} \begin{bmatrix} 11 & -25 \\ -4 & 9 \end{bmatrix} = \begin{bmatrix} -11 & 25 \\ 4 & -9 \end{bmatrix}$

and $(BA)^{-1} = \frac{1}{11(9)-20(5)} \begin{bmatrix} 9 & -5 \\ -20 & 11 \end{bmatrix} = \begin{bmatrix} -9 & 5 \\ 20 & -11 \end{bmatrix}$. Ok, we did that. The point comes in the later problems.

6.2 19. Using the results of 18, find $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$. What is the relationship among $(AB)^{-1}$, $(BA)^{-1}$, $A^{-1}B^{-1}$, and $B^{-1}A^{-1}$?

For this problem we only need to multiply, and compare, using the prior results. $A^{-1}B^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -9 & 5 \\ 20 & -11 \end{bmatrix}$, which is interesting, and $B^{-1}A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -11 & 25 \\ 4 & -9 \end{bmatrix}$, also familiar. Comparing different results in these two problems, we see $A^{-1}B^{-1} = (BA)^{-1}$ and $B^{-1}A^{-1} = (AB)^{-1}$. Perhaps surprising or unexpected?

6.2 22. Using matrices A and B of 18, and the results of that exercise, decide whether $(A+B)^{-1}$ is equal to $A^{-1} + B^{-1}$. What fact is this related to for real numbers and not matrices?

$A + B = \begin{bmatrix} 6 & 5 \\ 4 & 6 \end{bmatrix}$ and $(A+B)^{-1} = \frac{1}{6(6)-4(5)} \begin{bmatrix} 6 & -5 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} \frac{6}{16} & -\frac{5}{16} \\ -\frac{4}{16} & \frac{6}{16} \end{bmatrix}$.

On the other hand, $A^{-1} + B^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & 4 \end{bmatrix}$.

These are *very* different. Much like $\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$, e.g. $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, but $\frac{1}{2+3} = \frac{1}{5}$, also very different.