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0012 Demonstrate an understanding of the fundamental concepts of calculus.

I. Analyzing the concept of limit numerically, algebraically, graphically and in writing

A. Defining Limit

"The concept of limit is critical in the development of calculus. It revolves around the question: As x approach a certain number (or possibly +/- infinity) what behave does f(x) exhibit? "For a limit to exist, the limit from the left and the limit from the right must be equal" (Drost). The symbolism for limit is:

$$\lim_{x \to c} f(x) = L$$

This is read as "the limit as x approaches c of f(x) is L." This statement can still be true if f(c) does not equal L. The function does not have to be defined at c."

B. Analyzing limit numerically

"To construct the behavior of f(x) numerically, we construct a table of values." We let c be the value that x approaches and f(c) be the value mapped to by c.

Numerical Investigation

x	c-0.1	c-0.01	c-0.001 →	c	← c+0.001	c+0.01	c+0.1
f(x)	f(c)-0.1	f(c)-0.01	f(c)-0.001 →	f(c)	← f(c)+0.001	f(c)+0.01	f(c)+0.1

C. Analyzing limit algebraically

"Graphs are very useful tools for investigating limits, especially if something unusual happens at the point in question" (Bennett). Limits can be analyzed algebraically in the three following ways:

1. Direct Substitution

To analyze the limit of a function f(x) algebraically as x approaches a value c, we simply plug the value of c in for x. This method is called direct substitution. This method can only be used if a function is continuous at c and it works for limits of all polynomial functions. It also works for radical function (Randall).

2. Factor and Cancel

Another analytical method for finding the limit of f(x) is called "factor and cancel". This is used when direct substitution into a rational function leads to an undefined expression (i.e.with zero in the denominator).

3. Rationalize the numerator

When there is a radical in the numerator it is useful to rationalize the numerator. This can be done by multiplying the function by 1 (numerator/numerator).

D. Analyzing limit graphically

We can graphically analyze a limit by graphing the function. Using a graphing calculator allows us to zoom in closer and closer to c to assess the value of f(c).

II. Interpreting the derivative as the limit of the difference quotient

A. Defining the Difference Quotient

The difference quotient at x=c is the slope of a secant line that joins a point on the graph of f(c) with some nearby point, call it f(c+h). We call the limit of this difference quotient the first derivative of f(x) evaluated at x=c. This limit is actually the slope of the tangent line at x=c.

When the value of the derivative is positive, the graph of the tangent line has a positive slope and we say that the function is increasing. Conversely, when the value of the derivative is negative, the graph of the tangent line has a negative slope and we say that the function is decreasing. At any point where the value of the derivative is 0, the line tangent to the graph is horizontal and the function is neither increasing nor decreasing. (Austincc)

B. Formulas for the Difference Quotient

1. The [*]Difference Quotient, or the average rate of change of f(x) over the interval [a, b] is:

$$\frac{f}{x} = \frac{f(b)-f(a)}{b-a}$$

2. An alternative formula for the difference quotient replaces b with a+h:

$$\text{Average rate of change} = \frac{f(a+h) - f(a)}{h}$$

III. Interpreting the definite integral as the limit of a Riemann sum

A. Defining a Riemann sum

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[*] zweigmedia

[**] Ostebee and Zorn

A Riemann sum is a system for approximating the values of integrals.
"Let the interval [a,b] be partitioned into n subintervals by any n+1 points

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

and let the width of each subinterval be denoted by $\triangle x = x_i - x_{i-1}$ denote the width of the ith subinterval. Within each subinterval $[x_{i-1}, x_i]$, choose any sampling point c_i. The sum

$$S_n = f(c_1)\triangle x_1 + f(c_2)\triangle x_2 + \ldots + f(c_n)\triangle x_n$$

is a Riemann sum with n subdivision for f on [a,b]."**

B. Limit of Riemann sum as an interpretation of definite integral

Let a function f(x) be defined on the interval [a,b]. The integral of f over [a,b] is the number, if one exists, to which all Riemann sums S_n approach as n approaches infinity and as the widths of all subdivisions appoach zero. In symbols:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

IV. Applying the fundamental theorem of calculus

A. Definition

The first part of fundamental theorem of calculus states: Let f be continuous real-valued function defined on a closed interval [a,b], and F be the function defined for x in [a,b} by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

then, for every x in [a,b]:

$$F'(x) = f(x)$$

The second part of the fundamental theorem of calculus states that, if f is continuous on the closed interval [a,b] and F is the antiderivative (indefinite integral) of f on [a,b] for all x in [a,b]:

$$f(x) = F'(x)$$

then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

B. Application

1. The Fundamental Theorem of Calculus (FTC) is applied whenever an integral is evaluated. We apply this theorem whenever we wish to evaluate the derivative of a function between any two points. Physically, this theorem allows us to find the area under a function on a graph (or the area between the function and the x-axis, given the function lies below the x-axis).
2. The FTC is also applied when finding the area between two functions.
3. Lastly, the FTC is used when finding the volume of a solid revolution.

V. Applying the concepts of derivatives to interpret gradients, tangents, and slopes.

A. Gradients

If f is a function of two variables, then the gradient of f is:

$$\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j$$

The concept of derivatives often uses the gradient to find the directional derivative of f in the direction of the unit vector u by dotting the gradient of f with u.

$$\nabla f(x,y) = \nabla f(x,y) \cdot u$$

B. Tangents

"The slope of a curve at a point P can be thought of as the slope of a certain straight line through P, called the tangent line at P. Roughly speaking, the tangent line 'points in the direction of the curve' at P(Ostebee and Zorn, page 40)."

Let F'(x) be the derivative of F(x). Then:

1. F'(x)>0 implies the tangent line at F(x) is increasing at x
2. F'(x)=0 implies the tangent line at F(x) is constant
3. F'(x)<0 implies the tangent line at F(x) is decreasing at x.

C. Slopes

Let F'(x) be the derivative of F(x). Then:

4. F'(x)>0 implies F(x) has a positive slope at x
5. F'(x)=0 implies that x is a critical point for F(x) and the slope is 0 at x
6. F'(x)<0 implies that F(x) has a negative slope at x.

VI. Applying the concept of limit to analyze and interpret the properties of functions (e.g. continuity, asymptotes)

A. Asymptotes

An asymptote is a straight line or curve to which another curve approaches closer and closer when moving along it. (A curve may or may not actually touch or cross its asymptote. In fact, the curve may intersect the asymptote an infinite number of times.) Suppose *f* is a function. Then the line *y* = *a* is a **horizontal asymptote** for *f* if $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to-\infty} f(x) = a.$

This means that f(x) will get closer and closer to *a* depending on how big x is.

Note that if $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to-\infty} f(x) = b$

then the graph of *f* has two horizontal asymptotes: *y* = *a* and *y* = *b* (Example: the arctangent function).

The line *x* = *a* is a **vertical asymptote** of a function *f* if either of the following conditions is true:

$$\lim_{x\to a^-} f(x) = \pm\infty$$
$$\lim_{x\to a^+} f(x) = \pm\infty$$

B. Continuity

A function f is continuous at a number *a* if the following three conditions are satisfied:

(i) f is defined on an open interval containing *a*
(ii) the limit of f(x) as x approaches *a* exists
(iii) the limit of f(x) as x approaches *a* equals f(*a*)

VII. Applying the concept of rate of change to interpret statements from science, technology, economics, and other disciplines

A. Science

In Physics, the derivative is often interpreted as the instantaneous rate of change of a function or the velocity. It is also used to assess the acceleration of an object. By using derivatives, one can also evaluate the reaction rate of a chemical reaction.

B. Technology

Rates of change of used in technology to find absolute maximums, minimums and for optimization. Derivatives can play an important role in investment strategies by enabling the most flexible and cost effective strategies. They are used to determine the most efficient ways to transport materials and design factories.

C. Economics

Rates of change are used in economics to evaluate exchange rates, outputs, money, and pricing.

Problems

1. Investigate the $\lim_{x\to-\infty} \int (2x^2 - 5)/(3x^2+x+2)$ algebraically.
2. Let f(x)=2x-5. Write the difference quotient and simplify it.
3. *(Riemann Sum)* Given the function $f(x) = x^3$, the closed bounded interval $[0, 1]$, and the partition $\left\{0, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, 1\right\}$, compute a Riemann sum.

4. Use the Fundamental Theorem of Calculus to evaluate:

$$\int_0^{\pi} 2\,e^x - 5\cos(x)\,dx$$

5. If $f(x,y)=x^2-4xy$
   (a) find the gradient of f at the point P(3,-2)
   (b) use the gradient to find the directional derivative of f at P(3,-2) in the direction from P(3,-2) to Q(4,1)
   (c) discuss the significance of the answer to part (b) if f(x,y) is the temperature at (x,y)
6. Let $f(x) = \frac{x^3 - 8}{x - 2}$ for $x \neq 2$. Show how to define $f(2)$ in order to make f a continuous function at 2.
7. A ball is dropped vertically from a height of 100 meters. Assume that the acceleration due to gravity has a magnitude of 10 m/s^2. What is the velocity at t=2?
8. Find any horizontal asymptotes for $f(x) = \frac{x}{x-3}$

Solutions

1. Since we are interested only in negative values of x, we may assume that x≠0. First we divide numerator and denominator of the given expression by x^2 and then employ the appropriate limit theorems. Thus

$$\lim_{x\to-\infty} \int (2x^2 - 5)/(3x^2+x+2)$$
$$= \lim_{x\to-\infty} \int ((2-(5/x)/3+(1/x)+(2/x^2))$$
$$= \lim_{x\to-\infty} \int ((2-(5/x))^2)/ \lim_{x\to-\infty} \int (3+(1/x)+(2/x^2))$$
$$=(2-0)/(3+0+0)$$
$$=2/3$$

2. (f(x+h)-f(x))/h = (2(x+h)-5-(2x-5)/h
= (2x+2h-5-2x+5)/h
=2h/2
=2

3. Organizing into a table we compute the values,

i	subinterval	x_i^*	$f(x_i^*)$	Δx_i
1	$[0, \frac{1}{2}]$	$\frac{1}{3}$	$(\frac{1}{3})^3 = \frac{1}{27}$	$\frac{1}{2} - 0 = \frac{1}{2}$
2	$[\frac{1}{2}, \frac{3}{4}]$	$\frac{2}{3}$	$(\frac{2}{3})^3 = \frac{8}{27}$	$\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$
3	$[\frac{3}{4}, \frac{5}{6}]$	$\frac{8}{10}$	$(\frac{8}{10})^3 = \frac{64}{125}$	$\frac{5}{6} - \frac{3}{4} = \frac{1}{12}$
4	$[\frac{5}{6}, 1]$	$\frac{9}{10}$	$(\frac{9}{10})^3 = \frac{729}{1000}$	$1 - \frac{5}{6} = \frac{1}{6}$

So the Riemann sum for these x_i^* is

$$\sum_{k=1}^{4} f(x_i^*)\,\Delta x_i$$

$$= f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + f(x_3^*)\,\Delta x_3 + f(x_4^*)\,\Delta x_4$$

$$= \left(\frac{1}{27}\right)\left(\frac{1}{2}\right) + \left(\frac{8}{27}\right)\left(\frac{1}{4}\right) + \left(\frac{64}{125}\right)\left(\frac{1}{12}\right) + \left(\frac{729}{1000}\right)\left(\frac{1}{6}\right)$$

$$= \frac{2773}{10800}.$$

4. $\int_0^{\Pi} 2e^x\text{-}5\cos(x)dx = \int_0^{\Pi} 2e^x dx + \int_0^{\Pi} -5x\cos(x)dx$

$=2e^x[e^{\Pi} - e^0] – 5[\sin(\Pi)\text{-}\sin(0)]$

$=2e^{\Pi} - 2$

5. (a) $\nabla f(x,y) = (2x-4y)i - 4xj$

Hence, P(3,-2),

$$\nabla f(3,-2) = (6+8)i - 12j = 14i - 12j$$

(b) If we let a= $\overrightarrow{PQ}$, then

a=<4-3,1-(-2)>=<1,3>=i+3j

A unit vector having the direction of $\overrightarrow{PQ}$ is

$$u = \frac{1}{|a|}a = \frac{1}{\sqrt{10}}(i+3j)$$

We then have $D_u f(3,-2) = \nabla f(3,-2) \bullet u$

$$= (14i-12j) \bullet \frac{1}{\sqrt{10}}(i+3j)$$

$$= \frac{1}{\sqrt{10}}(14-36) = -\frac{22}{\sqrt{10}} \approx -6.96$$

(c)If f(x,y) is the temperature (in degrees) at (x,y), then the fact that $D_u f(-3,2) \approx -6.96$ indicates that if a point moves in the direction of $\overrightarrow{PQ}$,the temperature at P is decreasing at approximately 6.96 degrees per unit change in distance.

Let P(x,y) be a fixed point, and consider the directional derivative $D_u f(x,y)$ as $u=<u_1,u_2>$ varies. For certain unit vectors u the directional derivative may be positive (that is, f(x,y) may increase), or negative (f(x,y) may decrease), or it may be 0. In many applications it is important to find the direction in which f(x,y) increases most rapidly and also to find the maximum rate of change.

6. We have

$$\frac{x^3-8}{x-2}$$

is equal to…

$$\frac{(x-2)(x^2+2x+4)}{(x-2)}$$

which is equal to…

$(x^2 + 2x + 4)$

Thus $f(x) \rightarrow (2^2 + 2.2 + 4) = 12$ as $x \rightarrow 2$. So defining $f(2) = 12$ makes f continuous at 2, (and hence for all values of x).

7. Here, acceleration is in the negative direction, since it is toward the ground, which is designated with a coordinate of zero. So, $a = -10$ meters per second per second, and $v(t) = v_0 + a(t)$. But the initial velocity is zero, so $v(t) = -10t$. $v(2) = -20$ meters per second.

8. To find horizontal asymptotes, we take limits at infinity:

$$\lim_{x\to\infty}\frac{x}{x-3} = \lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{x}{x}-\frac{3}{x}} = \lim_{x\to\infty}\frac{1}{1-\frac{3}{x}} = 1$$

$$\lim_{x\to-\infty}\frac{x}{x-3} = \lim_{x\to-\infty}\frac{\frac{x}{x}}{\frac{x}{x}-\frac{3}{x}} = \lim_{x\to-\infty}\frac{1}{1-\frac{3}{x}} = 1$$

So, the line $y = 1$ is a horizontal asymptote of this function at both positive and negative infinity.

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