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#16 Understand the principles and properties of coordinate geometry.

I. *Applying the principles of distance, midpoint, slope, parallelism, and perpendicularity to characterize coordinate geometric relationships.*

Recall that given two points (x_1, y_1) and (x_2, y_2):

Distance = $\sqrt{\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)}$

Midpoint = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Slope = $\frac{y_2 - y_1}{x_2 - x_1}$

Parallel – two lines are parallel if their *slopes* are congruent.
Perpendicular – two lines are perpendicular if their slopes are negative reciprocals of each other.

II. *Analyzing and applying transformations in the coordinate plane.*

When applying transformations to the shape in question, remember that you are going to apply the transformation to each individual point. The follow transformations are performed on the given point (x_1, y_1).

Reflections – **x-axis**: $(x_1, -y_1)$
y-axis: $(-x_1, y_1)$
origin: $(-x_1, -y_1)$
line y = x: (y_1, x_1)
line y = -x: $(-y_1, -x_1)$
over any line: preserve the distance from the point and line of reflection. "In other words, the line of reflection lies directly in the middle between the figure and its image."[1]

Translations: note that a translation is a "slide" so we are moving the point or object over a certain amount.

$$\mathrm{T}_{a,b} = (x_1 + a, y_1 + b)$$

Rotations - $R_{90} = (-y_1, x_1)$
$R_{180} = (-x_1, -y_1)$
$R_{270} = (y_1, -x_1)$
$R_{360} = (x_1, y_1)$, of course.[2]

Dilations: represent an expansion or contraction of an image. The amount of change is the scale factor, which the values of the point are multiplied by to yield the resulting point.

$$D_k = (kx_1, ky_1)$$

III. *Using coordinate geometry to prove theorems about geometric figures*

When writing a coordinate geometry proof, be sure to draw and label the graph, state the formulas you will be using, and show all work. You will be working with the points given and use notions of distance, slope, midpoint, parallelism, and perpendicularity to prove what is asked. In a sense you will use everything from the first two parts here.[3]

Triangles
- Isosceles – show two sides are congruent
- Right – show a right angle or fulfills the Pythagorean Theorem.
- Equilateral – show all three sides are congruent.

Parallelogram
- Both pairs of opposite sides parallel.
- Both pairs of opposite sides are congruent.
- One pair of opposite sides is congruent and parallel.
- Diagonals bisect each other.

Rectangle
- A parallelogram with one right angle.
- A parallelogram with congruent diagonals.

Rhombus
- An equilateral quadrilateral.
- A parallelogram with two consecutive congruent sides.
- A parallelogram with perpendicular diagonals.

Square
- A rectangle with two consecutive congruent sides.
- A rhombus with right angles.

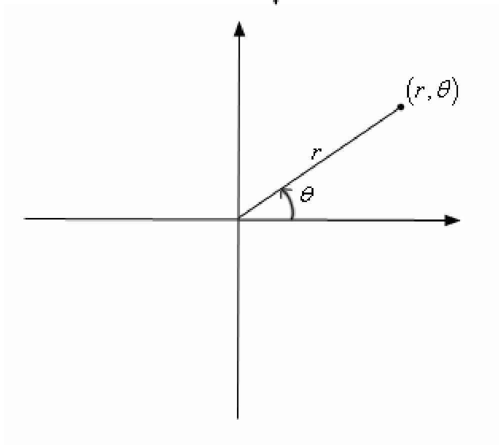
Trapezoid
- One pair of opposite sides parallel, along with the other pair not parallel.

Isosceles trapezoid
- A trapezoid with congruent legs.
- A trapezoid with congruent.[4]

IV. *Representing two and three dimensional geometric figure in various coordinate systems.*

Quick review of polar coordinates:[5]

$$x = r\cos\theta$$
$$y = r\sin\theta,$$
$$r = \sqrt{x^2 + y^2}$$



Let's go over some transformations on x^2. These transformations can be used in other conics as well.

1. $y = ax^2$ - a>1 is a vertical shrink by a factor of "a"
   0 < a < 1 is a vertical stretch by a factor of "a"
2. $y = ax^2$ - a<0, reflection over the x-axis.
3. $y = x^2 + k$ - Vertical Shift or slide
   k > 0: Vertical shift up k units
   k < 0: Vertical shift down $|k|$ units.
4. $y = (x - h)^2$ - Horizontal Shift or slide
   h > 0: Horizontal shift to the right h units.
   h < 0: Horizontal shift to the left $|h|$ units.

The order to do these transformations: Stretch/Shrink, Reflect, Shift, Shift.
When looking at a standard form of the parabola we can see different ways we can transform it. Also, we can get quadratic expression in this form by completing the square.

$$y = a(x-h)^2 + k$$ [6]

V. *Applying the distance formula to derive the equation of a conic section.*

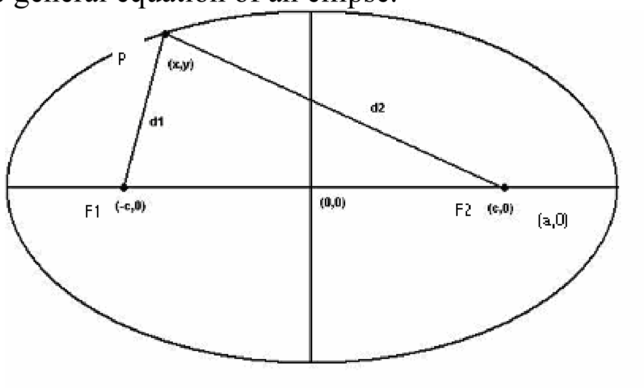
Distance = $\sqrt{\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)}$

We can find the standard form of the equation of a circle by squaring each side of the equation, where r is the radius of the circle and (h,k) is the center.

$\sqrt{\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)} = \text{r}$

$(x-h)^2 + (y-k)^2 = r^2$ - Standard form of circle equation.

Next we will derive the general equation of an ellipse.



By definition $PF_1 + PF_2 = 2a$ and $a^2 - c^2 = b^2$

Thus using the distance formula we can substitute in.

- $\sqrt{\left((x+c)^2 + y^2\right)} + \sqrt{\left((x-c)^2 + y^2\right)} = 2a$

Moving the $\sqrt{\left((x-c)^2 + y^2\right)}$ term we get:

- $(x+c)^2 + y^2 = 4a^2 - 4a * \sqrt{\left((x-c)^2 + y^2\right)} + (x-c)^2 + y^2$
- $x^2 + 2xc + c^2 + y^2 = 4a^2 + 4a * \sqrt{\left((x-c)^2 + y^2\right)} + x^2 - 2cx + c^2 + y^2$

Using algebra we can break it down to:

- $cx - a^2 = -a * \sqrt{\left((x-c)^2 + y^2\right)}$

Squaring both sides:

- $c^2x^2 - 2a^2cx + a^4 = a^2[(x-c)^2 + y^2]$

After more algebra and gathering variables we get:

- $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

But since $a^2 - c^2 = b^2$, we can substitute and divide each side by a^2b^2 to get the standard form of an ellipse:

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Use this standard form to construct ellipses. Where 2a is the length of the horizontal axis and 2b is the length of the vertical axis, with (h,k) the center:

- $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ [7]

VI. *Modeling and solving problems using conic sections.*

only one nappe of the cone (see (c)). If the plane intersects both nappes, a **hyperbola** is formed (see (d)).

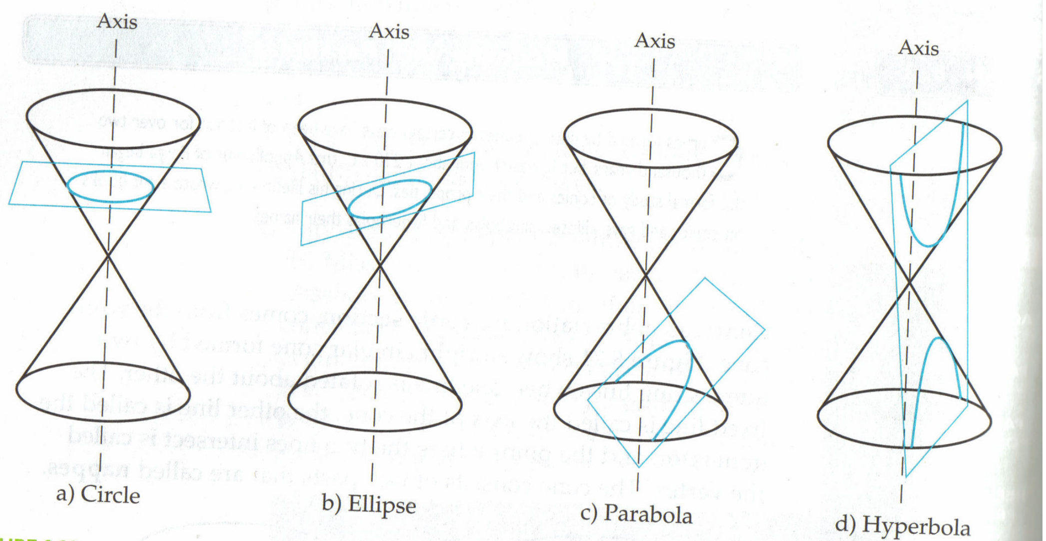


FIGURE 8.22.

"Conics are curves that come from the intersection of a right circular cone and a plane…If the plane is perpendicular to the axis of the cone the curve is a circle. If the plane is oblique to the axis and intersects all edges of one part of the cone, the curve is an ellipse. A parabola is formed when the plane is tilted so that it is parallel to one generation of the cone only one part of the cone. If the plain intersects both parts, it is a hyperbola." [8]

For these types of problems there will be a real life example like a flashlight, the one you will see shortly, and you will have to determine which type of conic it is. The following is the general form for a circle, parabola, ellipse, and hyperbola. Of which, parabola, hyperbola, and ellipse are on the given formula sheet for the exam.

Circle: $(x-h)^2 + (y-k)^2 = r^2$

Parabola: $(y-k)^2 = 4c(x-h)$

Ellipse: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Hyperbola: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

Examples:

1. The vertices of quadrilateral DEFG are D (3,2), E (7,4), F (9,8), and G (5,6). Using coordinate geometry, prove that
   a. DF and GE bisect each other.
   b. DEFG is a rhombus.[9]

2. Triangle CAT has vertices C (-2,6), A (6,4), and T (0,-2), and points S (3,1) is on side AT. Prove that
   a. CAT is isosceles
   b. CS is the perpendicular bisector of AT.[10]

3. Show the following
   a. Draw the graph of circle O with equation $(x-1)^2+(y+3)^2=16$
   b. Draw the image of O after the translation $T_{(-2,4)}$ and label it O'
   c. Write the equation of circle O'
   d. Write an equation of the line that passes through the centers of circle O and circle O'.[11]

4. Quadrilateral DRAW has vertices D (-3,6), R (6,3), A (6,-2), and W (-6,2). Prove the quadrilateral DRAW is an isosceles trapezoid.[12]

5. The coordinates of the endpoints of line segment ME are M (-4,-1) and E (5,4).
   a. Segment ST is the image of ME after the transformation $T_{(2,-6)}$, where S is the image of M, and T is the image of E. Construct both ME and ST.
   b. Is the quadrilateral METS a parallelogram?[13]

6. A flashlight has a circular face and is tilted at an acute angle to create a closed illuminated figure on a vertical wall as shown above. Which of the following equations could be used to represent the border of the figure if an (x,y) coordinate system is placed on the wall?
   a. $\frac{x^2}{a^2}+\frac{y}{b}=1$
   b. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
   c. $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$
   d. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ [14]

7. Triangle ABC is formed by the following points: A (3,-3), B (-1,-5), and C (5, -4). Draw the following images.

a. A'B'C', the image of ABC after the rotation R_{90°.

b. A"B"C", the image of A'B'C' after the translation $T_{(-4,-1)}$

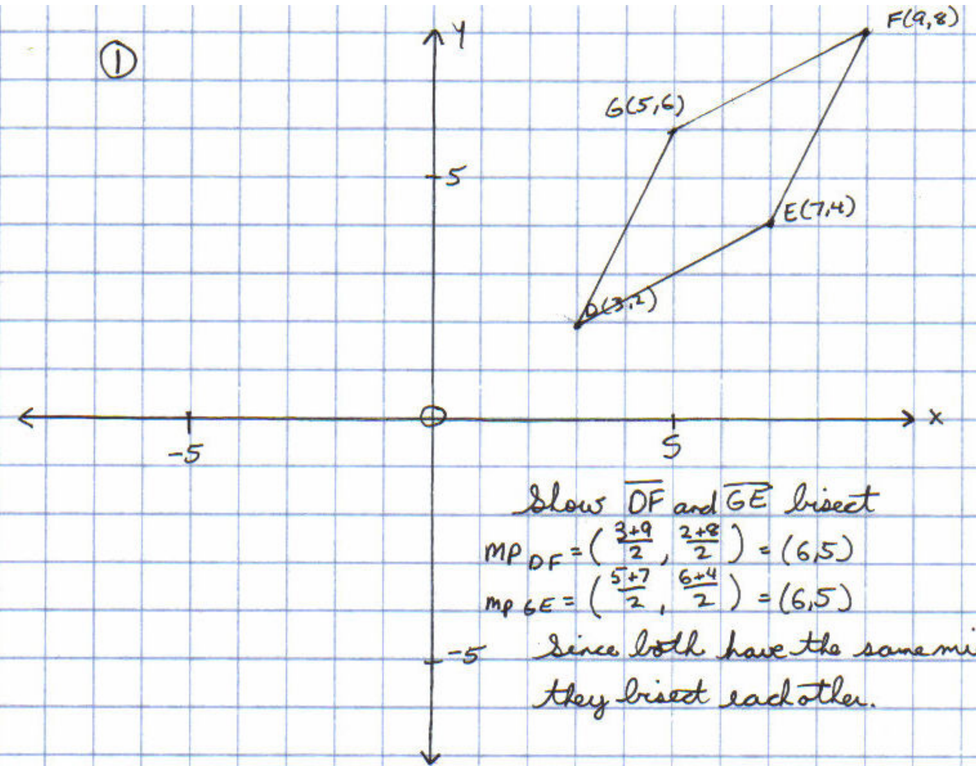
c. A"'B"'C"', the image of A"B"C" after the dilation D_3

d. Is the complete transformation an isometry?[15]

8. Describe and graph the following parabola $y = 4(x-2)^2 + 1$

Solutions to follow:

①



Show $\overline{DF}$ and $\overline{GE}$ bisect

$mp_{DF} = \left(\frac{3+9}{2}, \frac{2+8}{2}\right) = (6,5)$

$mp_{GE} = \left(\frac{5+7}{2}, \frac{6+4}{2}\right) = (6,5)$

Since both have the same midpoint, they bisect each other.

Show DEFG is a rhombus:

$m_{GF} = \frac{8-6}{9-5} = \frac{1}{2}$ $\quad m_{GF} = m_{DE}$

$m_{DE} = \frac{4-2}{7-3} = \frac{1}{2}$ $\quad \therefore$ parallel

$m_{GD} = \frac{6-2}{5-3} = 2$ $\quad m_{GD} = m_{EF}$

$m_{EF} = \frac{8-4}{9-7} = 2$ $\quad \therefore$ parallel

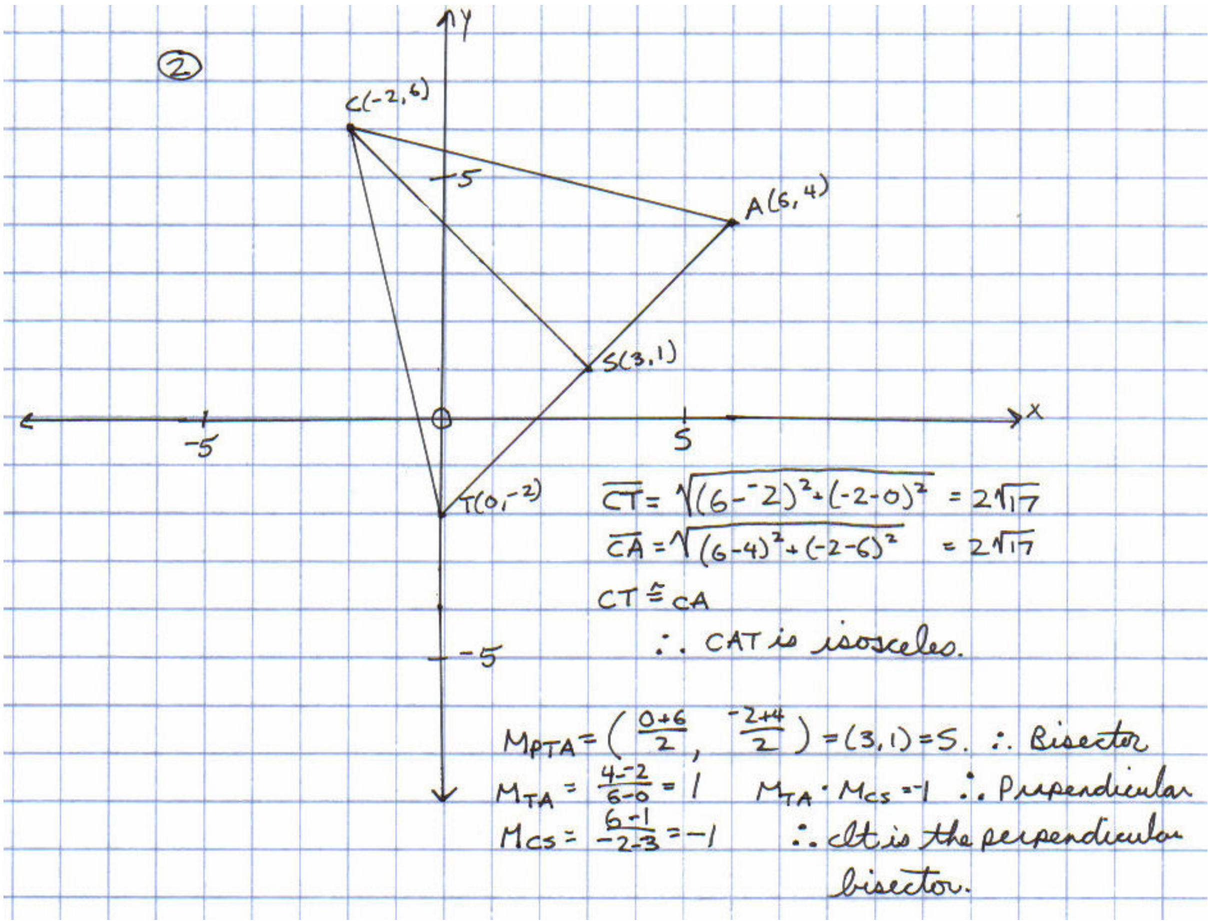
Two pairs of parallel sides. $\therefore$ Parallelogram

$m_{DF} = \frac{8-2}{9-3} = 1$ $\quad m_{DF} \cdot m_{GE} = -1$

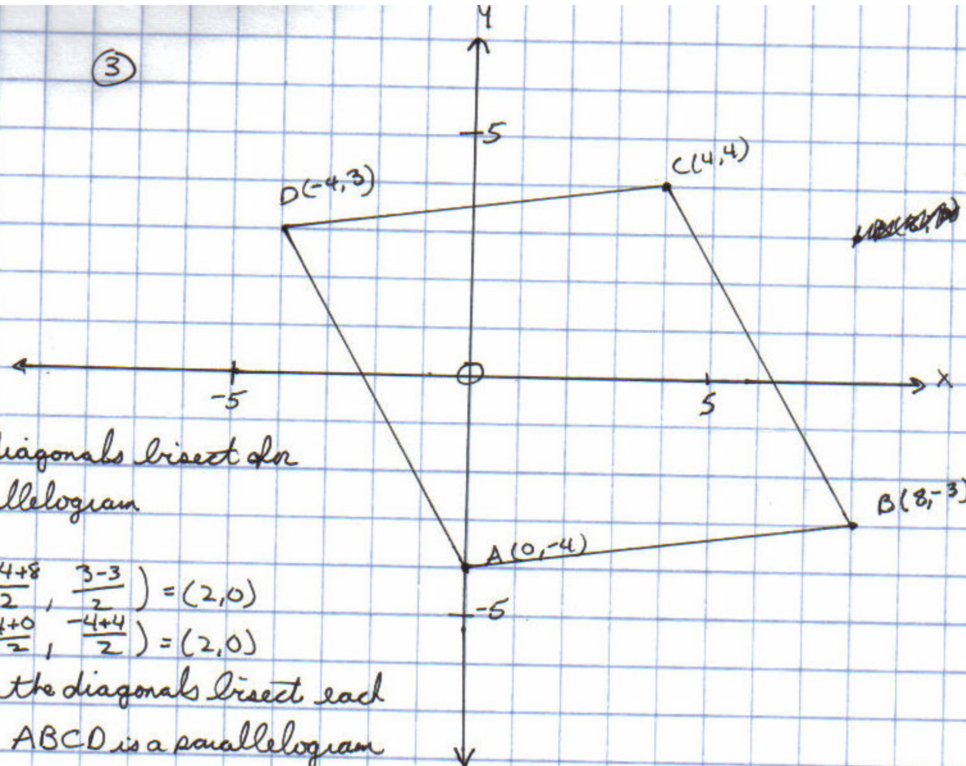
$m_{GE} = \frac{6-4}{5-7} = -1$ $\quad \therefore$ Form a right angle.

Since the diagonals form a right angle and it is a parallelogram,

DEFG is a rhombus.


②
y
x
C(-2,6)
A(6,4)
S(3,1)
T(0,-2)
5
-5
5
-5
CT = √((6 - -2)² + (-2-0)²) = 2√17
CA = √((6-4)² + (-2-6)²) = 2√17
CT ≅ CA
∴ CAT is isosceles.
M_PTA = ((0+6)/2, (-2+4)/2) = (3,1) = S. ∴ Bisector
M_TA = (4 - -2)/(6-0) = 1
M_TA · M_CS = -1 ∴ Perpendicular
M_CS = (6-1)/(-2-3) = -1
∴ It is the perpendicular bisector.

(3)



Show diagonals bisect for a parallelogram

$MP_{DB} = \left(\frac{-4+8}{2}, \frac{3-3}{2}\right) = (2,0)$

$MP_{AC} = \left(\frac{4+0}{2}, \frac{-4+4}{2}\right) = (2,0)$

Since the diagonals bisect each other, ABCD is a parallelogram

Show 2 consecutive congruent sides for rhombus

$\overline{DA} = \sqrt{(3 - -4)^2 + (-4-0)^2} = \sqrt{65}$

$\overline{AB} = \sqrt{(-3 - -4)^2 + (8-0)^2} = \sqrt{65}$

> two consecutive congruent sides.

A parallelogram with 2 consecutive sides is a rhombus.

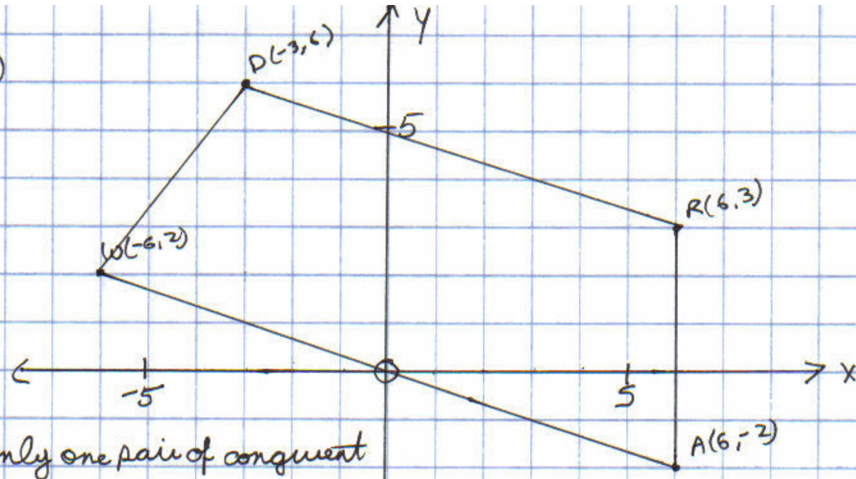
Show not a right angle.

$m_{AB} = \frac{-4 - -3}{0-8} = \frac{1}{8}$

$m_{DA} = \frac{3 - -4}{-4-0} = -\frac{7}{4}$

> not negative reciprocal ∴ <u>not</u> a rectangle

④



Show only one pair of congruent sides.

$m_{DR} = \frac{6-3}{-3-6} = \frac{-1}{3}$

$m_{WA} = \frac{2-^-2}{^-6-6} = \frac{-1}{3}$

$m_{DR} \parallel m_{WA}$.

$m_{DW} = \frac{6-2}{-3-^-6} = \frac{4}{3}$ } certainly not parallel

$m_{RA} = \frac{3-^-2}{6-6}$ = undef

There is only one pair of parallel side

DRAW is a trapezoid

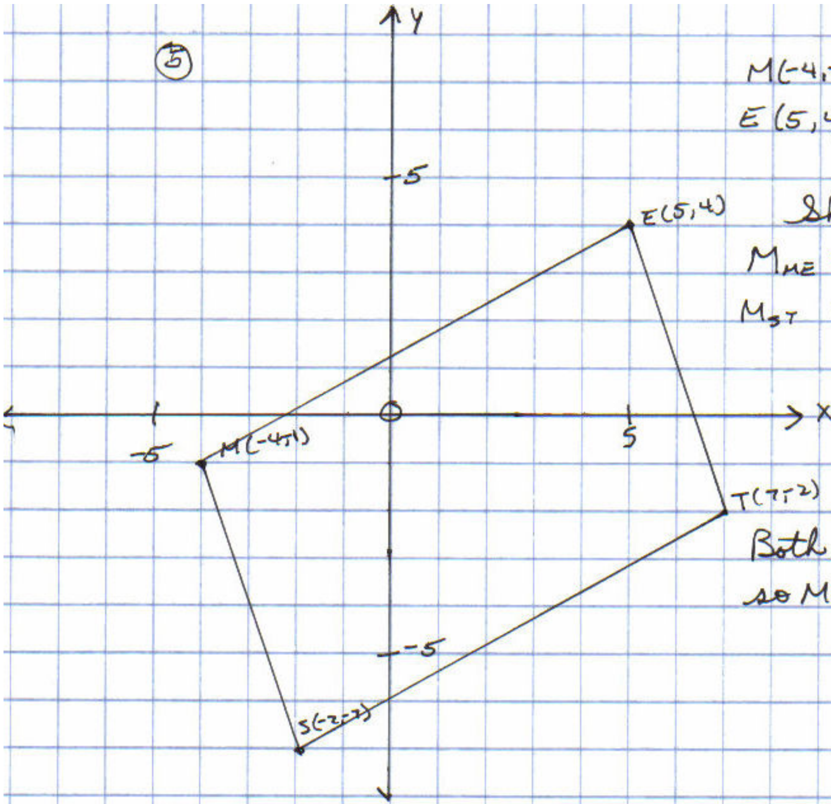
Show the legs are congruent

$\overline{DW} = \sqrt{(6-2)^2+(-3-^-6)^2} = 5$

$\overline{RA} = \sqrt{(3-^-2)^2+(6-6)^2} = 5$

Since the legs are congruent on a trapezoid, it is isosceles

⑤



$T_{2,-6}$

M(-4,-1) → S(-2,-7)

E(5,4) → T(7,-2)

Show parallelogram.

$M_{ME} = \frac{-1-4}{-4-5} = \frac{5}{9}$
$M_{ST} = \frac{-2 - -7}{7 - -2} = \frac{5}{9}$ > parallel

$M_{MS} = \frac{-1 - -7}{-4 - -2} = -3$
$M_{ET} = \frac{4 - -2}{5 - 7} = -3$ > parallel

Both pairs of sides are congruent so METS is a parallelogram.

⑥ A flashlight has a circular face...

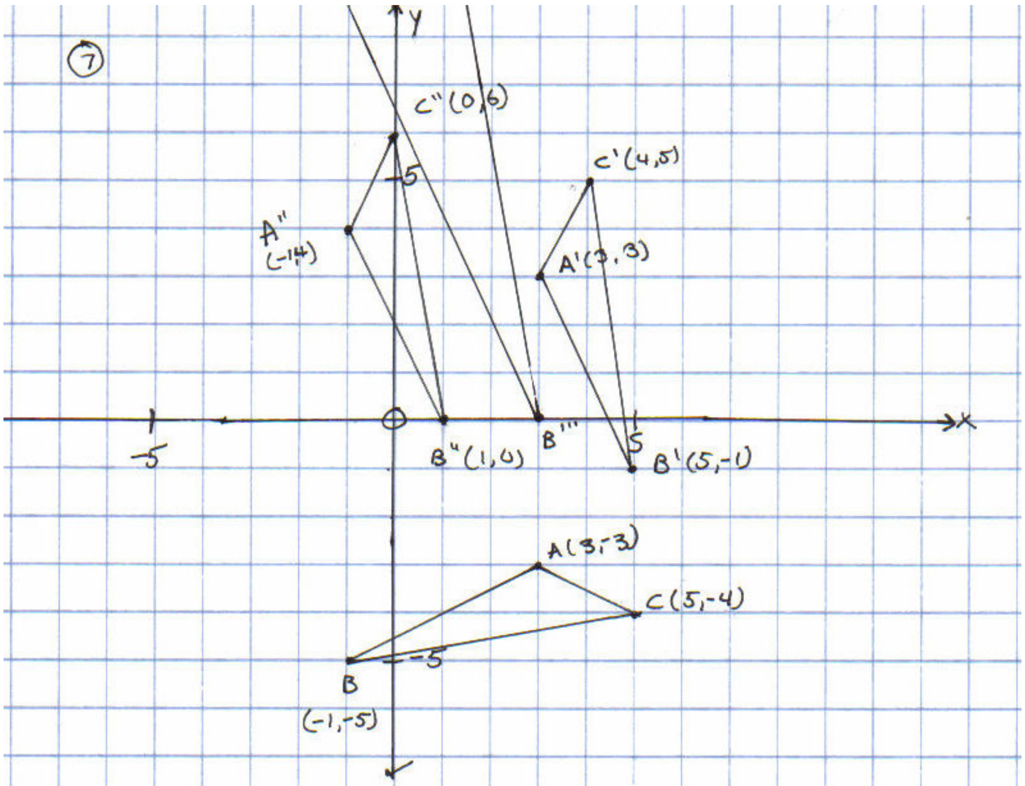
The flashlight creates a cone of light, and the intersection of the wall (a plane) and the cone is a conic section. So we must think of ellipse.

Since the angle is acute the conic section is an ellipse.

Looking at the chooses available (B) is the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

⑦



$A(3,-3) \xrightarrow{R_{90°}\,(-y,x)} A'(3,3) \xrightarrow{T_{-4,1}} A''(-1,4) \xrightarrow{D_3} A'''(-3,12)$

$B(-1,-5) \longrightarrow B'(5,-1) \longrightarrow B''(1,0) \longrightarrow B'''(3,0)$

$C(5,-4) \longrightarrow C'(4,5) \longrightarrow C''(0,6) \longrightarrow C'''(0,18)$

It is not an isometry because distance was not preserved. (Dilation)

⑧ $y = 4(x-2)^2+1$

- a Vertical shrink by a factor of 4
- No reflection of the x-axis
- Horizontal Shift to the right by a factor of 2.
- Vertical Shift up 1.

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[1] http://regentsprep.org/Regents/math/math-a.cfm
[2] Course III notes
[3] http://regentsprep.org/Regents/mathb/1D/Coordinatelesson.htm
[4] Course II handout

[5] http://en.wikipedia.org/wiki/Polar_coordinate_system
[6] Pre-Calculus notes
[7] Solomon Garfunkel. *Precalculus: Modeling our World*. (W.H. Freeman and Company: New York, 2001), 511.
[8] Ibid, 490.
[9] Course II, August 1999, #42.
[10] Course II, January 2000, #42.
[11] Course II, August 1998, #40.
[12] Course II, June 1998, #42.
[13] Course II, June 1999 #38.
[14] CST Prep Guide
[15] Course III June 1999, #38.