Mathmatics 221 Daily Problems

§2.1 39 The hiker reachest the top (or at least the end of his climb) at shortly before 2 hours. Just before reaching the top he was going the fastest - apparently he was excited to get there. At about $3\frac{1}{3}$ hours he was going fastest on the way down. When the graph is level, he stopped. Probably he took a break on the way up, and then stopped at the top to admire the view before heading back down.

§1.2 33 Let $f(x) = \frac{x^2}{x}$. $\lim_{x\to 0} f(x) = 0$, but $f(0)$ does not exist. On the other hand, let $g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$. $g(0) = 1$, but $\lim_{x\to 0} g(x)$ does not exist as $\lim_{x\to 0^+} g(x) = 1$ and $\lim_{x\to 0^-} g(x) = -1$

§1.2 35 This was answered in §1.6. One way to clarify it is that we should be able to get as close to the limit as we want. In particular, x^2 never gets under 0.01 away from -0.01, because it's always positive.

§1.3 61 The details here should be unsurprising. Start by reading the bottom on p. for the proof for $\lim_{x\to a}[f(x)]^2$. Here's the solution for 61:

$$\lim_{x\to a}[f(x)]^{n+1} = \lim_{x\to a}[f(x)^n \cdot f(x)] = \left[\lim_{x\to a} f(x)^n\right]\left[\lim_{xtoa} f(x)\right] = \left[\lim_{x\to a} f(x)^{n+1}\right]$$

§1.6 51 We wish to guarantee that $|2r^2 - 8| < \epsilon$. We have control over $|r-2|$. Consider our goal: $|2r^2 - 8| < \epsilon$. $|2r^2 - 8| = 2|r-2||r+2|$. Because we're talking about radii of washers we must know that $r \geq 0$. And we notice for such r, $|r+2| > |r-2|$. So, $|2r^2 - 8| < 2|r-2|^2$, and we thus want $2|r-2|^2 < \epsilon$. To do this we make $|r-2| < \sqrt{\frac{\epsilon}{2}} = \delta$, our response.

§2.2 23 See graph in class on 9/17.

§2.2 35 Average the slope for 1992 - 1994 and for 1994 - 1996 to get 0.4 as an estimate for 1994. (In fact they are the same). For 2000 we only have one end to use, so we use the slope from 1998 to 2000: 0.2. Cars are becoming more efficient, but less drastically (at least they were between 1994 and 2000).

§1.4 51 We want a linear function through both (30, 100) and (34, 0). g(T) = 850-25T satisfies this (use slopes, points &c).

§1.4 53 The function is $\begin{cases} x & \text{if } x < 100 \\ 80 & \text{if } x > 100 \end{cases}$. This is discontinuous at 100. This is the point when the box starts moving. Probably should be continuous. Connect with a very steep line there.

§2.3 51 The national debt is positive (as in the amount of debt is positive - let's not confuse this entire issue by thinking of how much surplus the nation has). This is $d(t) > 0$. Furthermore, the story seems to say that the debt is increasing, so $d(t)$ is increasing. Since it's increasing, it's derivative, or rate of change, is positive, $d'(t) > 0$. It says the "rate at

... increasing" is descreasing. That says that $d'(t)$, the rate of increasing, though positive, is decreasing. So, the derivative is decreasing. The derivative of the derivative, the second derivative, tells us how the derivative decreases. Since $d'(t)$ is decreasing, $d''(t) < 0$.

§2.6 41 Find the derivative of $\cos x$.

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} = \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$
$$= \cos(x)\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right) - \sin(x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) = \cos(x)(0) - \sin(x)(1) = -\sin(x)$$

§2.4 25 $Q'(t) = -0.04Q(t)$ - first negative because it's decreasing. Second - the rate of decrease is $0.04 = 4\%$ of the quantity sold, so $-0.04Q(t)$. $P'(t) = 0.03Q(t)$. So, $R'(t) = Q'(t)P(t) + Q(t)P'(t) = -0.04Q(t)P(t) + 0.03Q(t)P(t) = -0.01Q(t)P(t) = -0.01R(t)$. Therefore the revenue decreases at a rate of 1%. Mostly these two factors cancel each other out, and they would entirely if they were the same.

§2.4 32 alas 31 would have been much more interesting. $u'(v)$ is merely $\frac{0.2822}{0.217} \simeq 1.3$. This says that for an increase in club speed of 1 m/s, there will be an increase in ball speed of 1.3 m/s. Ball moves faster than club - sounds reasonable.

Let's try also looking at 31. $u'(m) = -\frac{14.11}{(m+0.05)^2}$. Because the denominator is always positive, and the rest has a negative in front, entirely it is negative. This says that as the mass of the club increases, the initial speed of the ball decreases. Probably why manufacturers want to make very light golf clubs.

§2.5 33 $h'(x) = f'(g(x))g'(x)$, so $h'(1) = f'(g(1))g'(1) = f'(2)g'(1) = 3(-2) = -6$.

§3.1 13 We want to estimate the amount sold at \$24 and \$36. For the first we'll use the tangent line at \$20 and for the second we'll use the tangent line at \$40. At \$20 the slope is -0.4 and at \$40 the slope is -0.2. So, the two tangent lines are $f(x) = -0.4(x-20)+18$ and $f(x) = -0.2(x-40)+12$. Using these respectively we find that $f(\$24) \doteq 16.4$ thousand games and $f(\$36) \doteq 12.8$ thousand games.

§2.7 23 Find the location of all horizontal and vertical tangents for $x^2 + y^3 - 3y = 4$. Differentiating implicitly, we get $2x + (3y^2-3)\frac{dy}{dx} = 0$. Not too much work to solve for the derivative: $\frac{dy}{dx} = \frac{2x}{3-3y^2}$. Horizontal tangents occur when the numerator is 0, which happens only for $x = 0$. Vertical tangents occur when the denominator is 0 which happens for $y = \pm 1$.

§3.7 28 We have the same similar triangles as in #27 (the right is similar to the whole) which still gives us $\frac{x+s}{18} = \frac{s}{6}$. To make things even easier, I'll clear fractions by multiplying by 18. This gives $x + s = 3s$, which could be even simpler as $x = 2s$. Ok, that's surely going to make things easy. We now differentiate both sides with respect to time to get $\frac{dx}{dt} = 2\frac{ds}{dt}$. As you see, there are no xs or ss in the new equation. So, we may ignore the fact that the person is 6 ft away from the lamppost. We need that they are walking toward the lamppost at a rate of 3 ft/s, this says that $\frac{dx}{dt} = -3$ ft/s. From the equation we found, we discover that $\frac{ds}{dt} = -\frac{3}{2}$ ft/s, which says that the shadow is shortening by 1.5 ft/s.

§3.7 51 - there is no such problem. Don't know what I was thinking. I guess I'm off easy on that one.

§3.2 33 $f(x) = \frac{3x^2}{x-3}$. I asked about absolute max and min, along with turning points. Turning points can be found by exploring where the derivative is zero or does not exist. The derivative (by the quotient rule) is $\frac{3x^2-18x}{(x-3)^2}$. This is undefined for $x = 3$ and equals zero for $x = 0$ and $x = 6$. So, for the first case, we work on the interval $[-2, 2]$. The only potential turning point on this interval is at $x = 0$, so we only need to consider $-2, 0, 2$. $f(-2) = -\frac{12}{5}$, $f(0) = 0$, and $f(2) = -12$, so the maximum is at 0 and the minimum is at 2.

In the second case, the situation is complicated. On $[2, 8]$ we also have $x = 3$ and $x = 6$ to deal with. $x = 6$ isn't a problem. $x = 3$ is a big problem. Not only is the derivative not defined there, the function is discontinuous there. So what we talked about today doesn't apply there. There's a vertical asymptote, which we'll discuss more about on Friday.

I'm convinced no one is reading these comments, and there remains a general lack of effort in this class. Prove me wrong. I will give 2 extra points on the next problem set to the first 3 people to email me saying you've seen this. I'll also do today's problem(s) believing there's still someone looking, but after that, I won't do any more until someone tells me they're reading here. Furthermore, I probably won't go back and do the ones I've missed.

§3.3 Here's what I would've done if we hadn't stalled on many precalculus topics:

Can a function with all negative values have a positive derivative? Always? The answer is "yes". First draw a graph with negative values which sometimes has positive derivative, then show one that always has positive derivative (for all real numbers, not just the ones you happen to see in your graph).

What must be true about a, b, c, d so that $ax^3 + bx^2 + cx + d$ is always increasing? The derivative must have no zeroes, and must be sometimes positive. $a > 0$ is good enough to make the derivative sometimes positive, as is $c > 0$. The first says that the derivative is an open-up parabola, the second says the parabola goes thru $y = c$ at $x = 0$. The derivative has zeroes at $\frac{-2b\pm\sqrt{4b^2-12ac}}{6a}$. For them to not exist, the quantity in the square root must be negative. So, the function is increasing if $b^2 < 3ac$. Notice that b^2 is non-negative, so $3ac > 0$, which means since one is positive, the other must be, too.

Ok, the actual problem here is §3.3 49 I want local extrema at ± 0.1. Working with the derivative we want turning points at those values. Using the suggestion $f'(x) = (x-0.1)(x+0.1)$ is negative between the two values and positive outside, so it does have the required sign changes. So, $f'(x) = x^2 - 0.01$. So, our next question is what is the function if this is the derivative. Well, let's do the last part first ... what has derivative equal to -0.01, well this should be obviously $-0.01x$. Now, what has derivative equal to x^2? Well, the exponent is reduced by one when we take derivatives, so it should be one *higher*.. Try x^3, but it's derivative is $3x^2$, so we're off by a factor of 3. Good solution - divide by 3. This gives us our function $f(x) = \frac{x^3}{3} - 0.01x$. Graph it. [That's not so obvious - they are difficult to find. A window that works well is $[-.2, .2] \times [-.002, .002]$.

Next it asks us to make a polynomial of degree 4 with two extrema near $x = 1$ and one near $x = 0$. Since the last one was tricky enough, let's not be too fancy. Let's make our turning points at $x = 0.1, 0.9, 1.1$, so $f'(x) = (x-0.1)(x-0.9)(x-1.1) = x^3 - 2.1x^2 + 1.19x - 0.099$. That's the derivative. By reasoning as above, a function which has this derivative is $f(x) = \frac{x^4}{4} - \frac{2.1x^3}{3} + \frac{1.19x^2}{2} - 0.099x$. Again, finding a good window is tricky. This is ok, but it's difficult to pull apart the two at 0.9 and 1.1: $[-.1, 1.3] \times [-.005, .07]$.