Lab 19: Becoming Secure with Sequences

Goals

- To begin thinking about the qualitative behavior of sequences.
- To gain familiarity with some important but non-obvious limits.
- To develop a feel for various rates of growth of sequences that diverge to infinity.

Before the Lab

Let $\{a_n\}$ be a sequence of real numbers. For the purposes of this lab, we distinguish four cases: If $\lim_{n\to\infty} a_n = \infty$, we say $\{a_n\}$ *diverges to infinity*; if $\lim_{n\to\infty} a_n = -\infty$, we say that $\{a_n\}$ *diverges to negative infinity*; if $\lim_{n\to\infty} a_n = c$, a finite real number, we say that $\{a_n\}$ *converges to* c; and in all other cases, we say that $\{a_n\}$ *diverges by oscillation*.

In this lab, you will gain insight into the behavior of $\{a_n\}$ by looking at the way the ordered pairs (n, a_n) lie when plotted in the plane.

1. Before you begin the lab, decide which of the four types of behavior applies to each of the following sequences.

   a. $a_n = \dfrac{n}{\sqrt{n}}$

   b. $a_n = \dfrac{n + (-1)^n}{n}$

   c. $a_n = \dfrac{2^n + n^3}{3^n + n^2}$

   d. $a_n = \dfrac{(-1)^n(n-1)}{n}$

In the lab

2. *A Friendly Warm-Up.* Use the computer to plot the ordered pairs (n, a_n) for the first 50 terms of each of the four sequences in Problem 1. You may need to adjust the horizontal scale to accommodate all fifty points on your screen, and you may need to adjust the vertical scale so that what you see is consistent with what you expected. Use the same graphical analysis to investigate the behavior of each of the following sequences. Be sure to keep track of the results.

   a. $a_n = \dfrac{\ln n}{\sqrt{n}}$

   b. $a_n = \dfrac{n^{10}}{2^n}$

   c. Experiment with several other values of k for sequences of the form $a_n = \dfrac{n^k}{2^n}$. Does changing the value of k seem to affect the value of the limit? As you change the value of the exponent k you may need to make changes in the vertical scale in order to see all the points of the sequence.

3. $\{r^n\}$ — *the complete story.* In Problem 2c you experimented with modifying the exponent. In this problem you will modify the base.

   a. Write the first few terms of the sequence $\{r^n\}$ for $r = 1$ and $r = -1$. (No need for the machine on this one.) Which does not converge?

   b. Determine the behavior of $\{r^n\}$ for other values of r, both positive and negative, keeping track of the behavior that you observe: divergent to infinity or negative infinity, convergent to a finite real number (which ?), or divergent by oscillation.

   c. Tell the whole story. In your written report for this lab, give a complete description of all possible types of behavior for the sequence $\{r^n\}$ as the value of r ranges over all real numbers. State clearly — with specific examples — which values of r give rise to which type of behavior.

4. *The race to infinity.* Most of the problems in this lab involve the relative rates at which sequences of positive terms diverge to infinity. We clarify the notion of "relative rates" in the following formal definition:

   Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers that diverge to infinity. We say that $\{a_n\}$ is *strictly faster* than $\{b_n\}$ if and only if $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, and that $\{a_n\}$ is *strictly slower* than $\{b_n\}$ if and only if $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$. If $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ is non-zero and finite, we say that $\{a_n\}$ is *comparable* to $\{b_n\}$.

On the basis of your earlier work in this lab, which is strictly faster, $\{\sqrt{n}\}$ or $\{\ln n\}$? Are $\{2^n\}$ and $\{n^2\}$ comparable? What about $\{2n-3\}$ and $\{3n-2\}$? Explain.

5. *Slowing things down.* In this problem, you will slow down the divergence to infinity of a sequence in two different ways: first by taking the logarithm and then by taking the n-th root of each term.

   a. $\{\ln a_n\}$: The sequence $\{2^n\}$ is strictly faster than $\{n\}$, but taking the logarithm of each of its terms results in a sequence that is comparable to n. (Why?) Find another sequence that is strictly faster than $\{n\}$ but which, when slowed down in this way, becomes strictly slower than $\{n\}$. Assemble graphical evidence to support your example.

   b. $\{a_n^{1/n}\}$: Taking the n-th root of the n-th term in the sequence $\{2^n\}$ results in the constant sequence $\{2\}$. (That's really putting on the brakes!) Discuss what happens to the sequence $\{n\}$ when you slow it down in this way. Does $\{n^{1/n}\}$ have a finite limit? What is it? Give graphical evidence to support your claim.

6. $\left\{\left(1+\frac{r}{n}\right)^n\right\}$— *variations on a theme.*

   a. It is a well-known and important fact that $\left\{\left(1+\frac{r}{n}\right)^n\right\}$ converges to e^r. By letting $r=1$, $\ln 2$, and $\ln 3$, assemble evidence in support of this fact.

   b. Investigate the behavior of two related sequences $\left\{\left(1+\frac{1}{n^2}\right)^n\right\}$ and $\left\{\left(1+\frac{1}{\sqrt{n}}\right)^n\right\}$. Discuss how the sequences differ in their behavior as n gets large. What do you think accounts for this difference?

Further Explorations

7. In Problem 6 you dealt with sequences of the type $\left\{\left(1+\frac{1}{a_n}\right)^{b_n}\right\}$ where both a_n and b_n diverged to infinity. In the first part of Problem 6, a_n and b_n diverged to infinity at the same rate; in the second part, they diverged to infinity at different rates. Try to find a precise connection between the limit of the sequence and the relative rates at which a_n and b_n diverge to infinity.

8. a. Apply the n-th root procedure to slow down the sequence $\{n!\}$. Graph the result. It still diverges to infinity, but at a slower rate. The points of the sequence should appear to lie along a straight line of constant slope.

   b. Use this to find a simpler sequence that is comparable to $\{n!^{1/n}\}$.

   c. Get an estimate for the numerical value of the slope. Look up Stirling's formula for $n!$ in your textbook or in a book of tables. Explain the connection between this formula and your estimate for the numerical value of the slope.