Lab 20: Getting Serious about Series

Goals

- To understand series convergence in terms of limits of partial sums.
- To explore the family of p-series.
- To investigate the growth of the partial sums for the harmonic series.
- To study the alternating harmonic series and what can happen when terms are rearranged.

In the Lab

In trying to attach meaning to an infinite series $\sum_{k=1}^{\infty} b_k$ of real numbers b_k, it is natural to study the behavior of the *partial sums* $S_N = \sum_{k=1}^{N} b_k$. If $\lim_{N\to\infty} S_N$ exists, we say that the series *converges* and we define the value of the limit to be the *sum* of the infinite series. Otherwise, we say the series *diverges*.

1. *A friendly warm-up.* Consider the geometric series $\sum_{k=0}^{\infty} r^k$, where the real number r gives the ratio between successive terms. Note that the series starts with $k = 0$ rather than $k = 1$.

   a. Let $r = \frac{4}{5}$ and use the computer to investigate the partial sums of the series $\sum_{k=0}^{\infty} (\frac{4}{5})^k$. As you evaluate partial sums with more and more terms, what appears to be the limiting value? Confirm that this value for the sum of the infinite series agrees with the well-known formula for the sum of a geometric series whose ratio r has absolute value less than 1.

   b. Let $r = 1.01$ and use the computer to study the partial sums. What do you conclude about the convergence or divergence of the geometric series in this case? Explain.

   c. Now let $r = 1$ and $r = -1$. Discuss the convergence or divergence of the series in these cases. Do you really need to use a computer in part c or even in part b? Explain.

d. When r is negative in a geometric series we get an example of an *alternating series*. Explain why this is an aptly chosen description. Give examples of two alternating geometric series, one convergent and one divergent.

e. Investigate the behavior of the geometric series for various values of the ratio r. Write a clear and complete statement summarizing your results.

2. *The p-series.* Now consider the *p-series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$, where p can be any positive real number.

a. For $p = .5$ and for $p = 2$, write the first five terms of the series in your notebook. Then by studying partial sums on the computer, argue that one of these series converges and one diverges. Do not get too ambitious here on the number of terms, N, in your partial sums; depending on your computer system, you may be in for an annoying wait. Give an estimate of the sum for the convergent series. [Obscure hint indicating a beautiful and mysterious fact: Multiply the partial sums by 6 and take the square root of the results.]

b. Select two more p values, one of which gives a convergent p-series and the other a divergent one. Give explanations and evidence for your conclusions.

c. When $p = 1$ we get a special and important case of the p-series known as the *harmonic series*, $\sum_{k=1}^{\infty} \frac{1}{k}$. Write out and then compute the sum of the first four terms of the harmonic series by hand. Then use the computer to estimate the partial sums for $N = 4$, $N = 128$, and $N = 1024$.

d. Based upon your results of parts a and b, formulate a preliminary conjecture about convergence and divergence of the p-series in terms of the positive real number p. Ignore the $p = 1$ case for now. That is the subject of the next problem.

3. *The harmonic series.* It is a remarkable and important fact that the harmonic series diverges. Were you persuaded otherwise by the fact that its terms tend to 0? Any straightforward attempt to compute this sum by a computer, no matter how large or powerful, will lead to the incorrect conclusion that the series converges. (See Problem 3e below.) In this problem we will use the computer to obtain a valuable hint about how to prove that the harmonic series diverges.

a. Building on what you did in Problem 2c, make a table of values for the partial sums of the harmonic series with $N = 4$, 8, 16, 32, 64, and 128. Confirm from your table that each time enough successive terms are added for N (the number of terms in the partial sum) to reach the next power of two, the partial sum increases by an amount exceeding $\frac{1}{2}$.

b. Assuming that this subtle but systematic process of increasing by $\frac{1}{2}$ persists for arbitrarily large N, argue, with no help from the computer, that the values of the partial sums will eventually exceed 14. Give a value of N for which the partial sum exceeds 14. Justify your answer.

c. Try to prove what you observed in part *a*: that by successively adding enough terms to any partial sum, we can further increase its value by at least $\frac{1}{2}$. Hint: Consider

$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}$$

and

$$\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}.$$

How many terms are in the first sum and what is the smallest term? What about the second sum? Finally, think similarly about the general case,

$$\frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}}.$$

Now put this all together in the clearest and most persuasive argument you can make, and you will have a proof that the harmonic series diverges.

d. Now, refine your answer to Problem 2*d* with a more definitive statement about convergence and divergence of p-series.

e. (Optional, but highly recommended for computer buffs) Write a paragraph or two attempting to justify, in your own words, the claim made above that any attempt to evaluate the harmonic series by direct numerical summation of terms on any computer using any amount of time will result in a finite sum and lead to the faulty conclusion that the series converges. Hint: The sum it "converges" to will depend on the precision being used in the calculation.

4. *The alternating harmonic series.*

a. If we change the signs of all the even numbered terms in the harmonic series, we get the *alternating harmonic series*, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. Write out the first four terms in the alternating harmonic series and compute the fourth partial sum by hand. Make sure you see how the $(-1)^{k+1}$ factor causes the signs to alternate.

b. We state without proof the fact that the alternating harmonic series converges. Use the computer to estimate the sum of this series. Hint: Compare your result with $\ln 2$.

Further Exploration

5. *Rearrangement of terms in the alternating harmonic series.*

   a. In many ways and in many situations, convergent infinite sums behave like finite ones. The infinite has its share of surprises, however, and one such surprise is that by suitably changing the order of the terms in the alternating harmonic series, the rearranged series can be made to converge to any real number we please. Consider the following rearrangement:

   $$1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12} + \cdots .$$

   Write the next six terms in the series to make sure you see the pattern. Use the computer to make your best guess about the sum of this series. Explain what you did. Can you confirm your result algebraically? Hint: Write the series as

   $$(1 - \tfrac{1}{2}) - \tfrac{1}{4} + (\tfrac{1}{3} - \tfrac{1}{6}) - \tfrac{1}{8} + \cdots$$

   and compare with the original alternating harmonic series.

   b. Find a rearrangement of terms of the alternating harmonic series that converges to a negative value. Explain your rearrangement rule and provide some evidence that the sum is indeed negative. The computer will help here, but think before you compute.

6. *Another alternating series.* Consider the alternating 2-series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$.

   Estimate the sum of this series. If terms in this alternating series are rearranged, does it seem to affect the sum? Justify your answer. Compare your results here with your results in Problem 5. Why do you think this alternating series is more "well-behaved" than the alternating harmonic series?

7. *A baseball joke.* What is $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\text{world}}}$?