222 §1.3 More Volumes and Shells

Ok, today we have one new idea and lots of variation on the ideas. Let's start with an example of the new idea. Consider the volume generated when the region bounded by $y = x^3 + x^2$, the x-interval $[0, 1]$, and the vertical line $x = 1$ is rotated around the y-axis. Notice, this one is fundamentally different from those we've done before. (1) To quickly review from last time, if this problem were to rotate around the x-axis, how would you set that up? But, it's not. So, how does that change things?

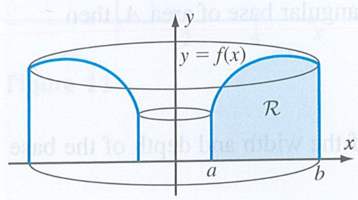


Figure 16

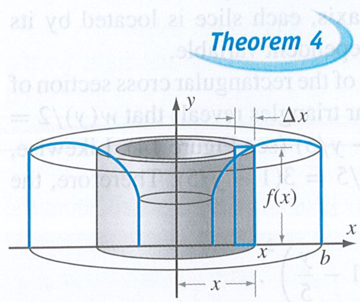
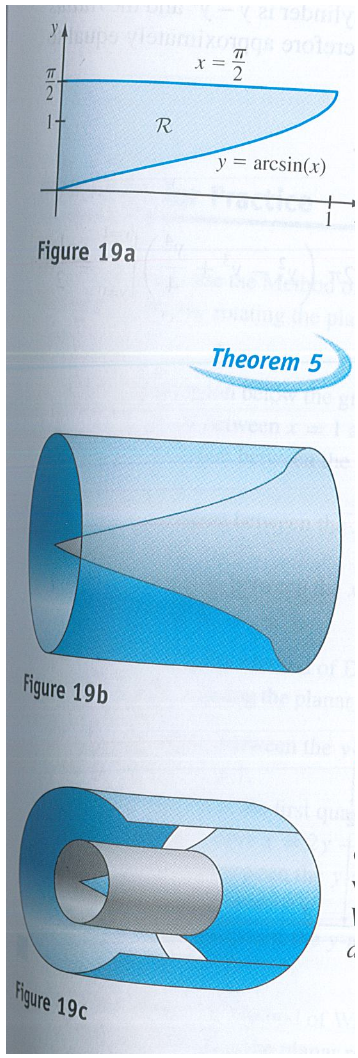


Figure 17

To get some view, start by taking just a narrow strip, a line segment above say, $x = \frac{1}{2}$ and rotating it around the y-axis. Think of rotating a pencil in the air around the y-axis. (2) What shape does it produce? It is an interesting shape that you see everyday, but you probably don't think about mathematically all that often. (3) What is the area of the sides of that shape, in general for a given radius r and height h? (If you don't recall this game from long ago in your education, the way to figure this out is to unroll it into a familiar shape and look for the dimensions.) If instead we think of a segment above an arbitrary x value, the radius is the distance from the y-axis that we are rotating around. The height is the distance above the x-axis. That is given by the function. (4) So, in terms of x, what is the area that is produced when rotating just one segment? Like we did yesterday, we use an integral to take the sum of each of them for different x values. (4) Putting all that together, what integral gives the volume when the region is rotated around the y-axis?

Let's look at two examples done both ways to see how this fits together, and something about why one would choose one way or another. Consider the region bounded above by the horizontal line $y = \frac{\pi}{2}$, below by the curve $y = \arcsin x$, $0 \leq x \leq 1$, and on the left by the y-axis. We will rotate this region around the x-axis. Since it is currently in terms of x, let's use vertical line segments first, one for each x value. So, think of a vertical line segment, its top is at $y = \frac{\pi}{2}$ and its bottom is at $y = \arcsin x$. And now spin it around the x-axis. (5) What shape does that make? (6) For a given x-value, what is the outer radius (it's constant, actually)? (7) For a given x-value, what is the inner radius (It's not constant). So, putting those together, (8) What is the cross-section area? (9) combine all this into an integral. Now, the point of this version ... (10) do you want to complete that integral? Why not? So ..., we try it the other way.



How? To work the other way around, we work in terms of y not x with horizontal instead of vertical line segments. This is our method for today. (11) So, if you have a horizontal line segment and rotate it around the x-axis, what shape do you get? (12) What is the distance from the axis, the radius, in terms of y? (hint: it's *very* simple.) To find the height, we'll nee to invert to make x in terms of y. I hope this is easy. (13) If $y = \arcsin x$, what is x in terms of y? (14) Putting this together what is the area of the cylinder? (15) And hence, what is the integral to find the volume? Now, a very real question (16) Which integral do you like better? Neither is obvious, I admit. Here's a true statement: *I* don't know how to do the first one, but we'll lear how to do the second early in Chapter 2, so ... stick around.

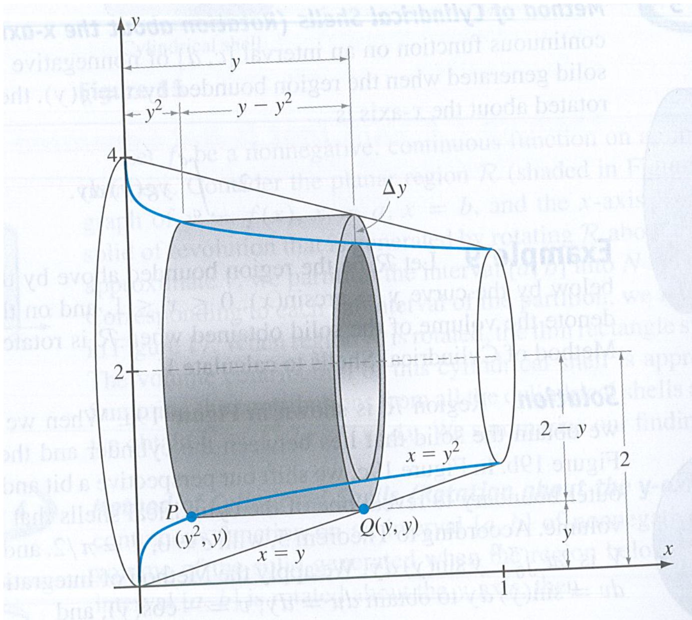


Figure 21

Ok, one more example done two ways. This time we can actually do both integrals, I think. We're going to look at the region between $x = y^2$ and $x = y$. It's a little sliver. But, this time we're not going to rotate around an axis. This time we will rotate around the horizontal line $y = 2$. That will change a few of the computations. To start (16) what are the x and y values where the two bounding curves intersect? Those will be our integration bounds. That's the easy part. We'll do this two ways. First we'll work in terms of x. To do so (17) Solve the first equation $x = y^2$ for y in terms of x. Technically you have two choices there, but one is the right one given the intersection you found before. Ok, now, take a vertical line segment, and rotate it around the horizontal line $y = 2$. (18) What shape does that produce? (19) When $x = 0$ what is the outer radius? (20) When $x = 1$ what is the outer radius? (21) What about when $x = 0.1$? If you answered the last one, you should be ready for (22) what is the outer radius in terms of x in general? (23) Now do the same thing for the inner radius. (using the answer to 17). (24) Put these pieces together to set up and compute this integral. At some point you will distribute and integrate term-by-term.

Finally, let's do this example in terms of y, still around the line $y = 2$. So, this time, since we will work in terms of y we consider a horizontal line segment and rotate it . (25) What shape does that produce? It should be our other option. Similar to (22) above, (26) what is the radius in terms of y in general? (27) what is the height of the line segment in terms of y? (hint: it should be the difference between two functions here) (28) What is the area of the shape made in (25) in terms of y? (29) Put the pieces together one final time to set up and compute this integral. Again you will need to distribute. (30) Check if your answer agrees with (24). (31) Which way do you like better? They are different.

Knowing both methods means that you can use whichever one works better for a particular example. Most importantly, do *not* only learn one method. For most examples one method is easier than the other. And for each method there are examples that are literally impossible.