And now for something completely different. Good news - we're done with the most abstract part of the course, just a few things remaining here.

It's been a while, so let's consider an application. For uncontrolled simple growth (like bacteria with plenty of room to grow), it is reasonable that the growth rate of a population would be proportional to the amount present at a given point. Suppose there are 1000 bacteria at the beginning of the experiment and 1500 an hour later. How large will the population be after 5 hours? When will it reach 10000? We need some new techniques to deal with this (notice these aren't questions for you yet).

Let's use variables P and t. From the beginning we are given $\frac{dP}{dt} \propto P(t)$. First we take the standard pre-calculus move - (1) what does proportional mean? So, $\frac{dP}{dt} = kP(t)$ for some k. Before we do anything else, it'll be good to have P only on one side, divide both sides by $P(t)$. Next, because we have derivatives, with respect to time, (2) integrate both sides with respect to time to undo the derivatives. So far you should have $\int \frac{1}{P}\frac{dP}{dt}dt = \int kdt$. The right integral is surely easy. The left is a little sneaky-looking: what we have here is a simple substitution setup. Letting $P = P(t)$ and so $dP = \frac{dP}{dt}dt$ we then get that the left side is just $\int \frac{1}{P}dP = \int kdt$. Next, (3) integrate both sides. Don't forget $+C$, but you only need it on one side, either is fine. (4) Use the facts that $P(0) = 1000$ and $P(1) = 1500$ to determine C and k. (5) Solve for $P(t)$, and then find $P(5)$ and when $P(t) = 10000$. You should get $P(t) = 1000e^{t\ln(1.5)}$, or $P(t) = 1000(1.5)^t$, with $P(5) = 7593.75$ and $P(5.67887358727) = 10000$ (exactly $\frac{\ln 10}{\ln 1.5}$).

Let's look at generalising this into a method. Suppose $\frac{dy}{dx} = g(y)f(x)$, we can then "separate variables" to get $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$ and then integrating with respect to x gives $\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx$. Notice that substitution produces $\int \frac{1}{g(y)}dy = \int f(x)dx$. It's as if we multiplied across the dx. We didn't, but that's what the chain rule and substitution do for us. In the future we will treat it as if we are multiplying, because we've learned that they have the same effect. We will do all that by just saying "separate variables". Please note this means divide, not separation by addition.

What about decline instead of growth? Let's consider penicillin in the bloodstream. The rate at which it leaves is proportional to the amount present. If 50 mg remain 7 hours after 450 mg is injected, when was 200 mg present? This should be almost identical. $\frac{dp}{dt} = -kp$. (6) Why negative? Separate variables to get $\int \frac{1}{p}dp = \int -kdt$. And proceed as above. (7) Finish the question to get $\frac{7\ln(4/9)}{ln(1/9)} \doteq 2.583491725$ hours.

Newton's law of cooling says that the rate of change of the temperature is proportional to the difference from the ambient temperature. Suppose you have just made some coffee using boiling water, and you want to know how long to wait until the room temperature (20C) cools the coffee to ideal drinking temperature (60C). We need one more bit of information, so we wait two minutes and measure the temperature and find that it is already down to 85C. Now how we have enough information to answer the question. One getting started question: (8) Given "the rate of change of the temperature is proportional to the difference from the ambient temperature", which direction? In other words, if the temperature of the coffee is above the ambient temperature, is the coffee temperature increasing or decreasing? What if the temperature of iced coffee is below the ambient temperature, is the coffee temperature then increasing or decreasing? What does this tell us? The sign of the proportionality constant. (9) Write the equation that you find from proportionality with positive constant of proportionality. I would expect this to be something like $\frac{dT}{dt} = k(A - T)$. (10) Solve this differential equation using separation of variables to find the function $T = 80(\frac{65}{80})^{2t} + 20$ and solve to get for $T = 60$, $t = \frac{2\ln 2}{\ln(80/65)} \doteq 6.67645252431$ minutes.

Notice that the methods we've discussed so far have a chance of working for functions of the form: $y'(x) = g(y)f(x)$, but not really for anything else. Even something as innocent looking as $y'(x) = x + y$ we have no idea how to solve. I guess this is why there are three whole classes here for dealing with differential equations. We will learn the day after next of other ways to analyse without solving them.