222 §7.5

Wrap up on lab 10: (1) do you have a proof for the $\frac{0}{0}$ case?

Aside from the $\frac{0}{0}$ case, there are several other indeterminate forms. The most similar is the $\frac{\infty}{\infty}$ case, which can be simply interpreted as merely exchanging numerators and denominators. Happily Bernoulli's rule works just as well here. As a simple example, try (2) $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

Indeterminate forms arise where there is a conflict in rules. The simplest of these to see is $0\cdot\pm\infty$. (3) What is 0 times anything? (4) What is $\pm\infty$ times anything? What happens when they combine? Any answer is possible - the rules no longer hold. Here is an example: $\lim_{x\to 0^+} x\ln x$. In this form, Bernoulli's rule does not apply. (5) What can be done to transform this into a quotient without changing its value? (6) Do that, and then apply Bernoulli's rule. Afterward you should have $\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$. Now at this point you might think to apply again since we are back to $\frac{\infty}{-\infty}$. Notice if you keep doing that, that this will only get worse and we will never find an answer. However, at this stage if you simplify algebra you should quickly see the answer 0. (7) Please do.

Here is another indeterminate form: what number when added to infinity produces infinity? This is the question $\infty-\infty$ and any number could be an answer. Here is an example: $\lim_{x\to 0}\frac{1}{\sin x}-\frac{1}{x}$. This form is not very workable. (8) How can we transform it into a form appropriate for Bernoulli's rule? (9) Please do so and apply the rule. After applying the rule you should find $\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}$. Notice however this is *still* a $\frac{0}{0}$ form. So, we can apply the rule one more time. (10) When you do, I hope you will find $\frac{0}{2}=0$.

The most intriguing and varied indeterminate forms are the exponential forms. The most important example of this appeared in lab 16. Let's explore this important example and then see how it generalises. The example appeared when we were looking at compound interest. With continuous compounding of an annual rate r we found the yearly rate to be $\lim_{n\to\infty}(1+\frac{r}{n})^n$. Notice that this goes to 1^∞. Also notice that one to any power is one, but any number to the infinite power is infinity. This gives us another indeterminate form, and we have work to do. Notice the main challenge is that this is not in a fraction form. And the worst part is the exponent. How do we deal with pesky exponents? Logarithms. How do we do this without changing the answer? We need to introduce a new variable and then use logarithms and remember to undo at the end. Let $y=\lim_{n\to\infty}(1+\frac{r}{n})^n$. Now we can take the logarithm of both sides to get $\ln y=\ln\lim_{n\to\infty}(1+\frac{r}{n})^n=\lim_{n\to\infty}\ln((1+\frac{r}{n})^n)$. This first move works because natural logarithm is a continuous function, so the limit of the function is equal to the function of the limit. Now for the fun part. (11) What can we do with that exponent inside of the logarithm? Once that is done we have an $\infty\cdot 0$ form, as we saw above. We need to transform it into a quotient. (12) Do so, and I hope you arrive at $\lim_{n\to\infty}\frac{\ln(1+\frac{r}{n})}{\frac{1}{n}}$. Notice this is a standard $\frac{0}{0}$ form. It's not simple, but we've been working for a while and I think we can handle it. (13) Differentiate numerator and denominator, remembering the chain rule in the numerator. In fact the chain rule is the key, because it produces something that cancels. (14) Cancel and I hope you are left with $\lim_{n\to\infty}\frac{r}{1+\frac{r}{n}}$. (15) Ok, now what does this limit go to? That is not our final answer. Remember all of this equals $\ln y$. So for the last step we have to undo and get our final answer. (16) Do this. Your answer should agree with the conjecture made in lab 16 (question 2d).

There are many more exponential indeterminate forms. (17) Explain the two rules that conflict to make each of these indeterminate: 0^0, ∞^0 and $1^{-\infty}$. (18) Which type are each of these limits: $\lim_{x\to\infty} x^{1/x}$, $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{-x}$, $\lim_{x\to 0^+} x^x$? (19) Complete two of these limits. If you wish, we can try to work on the third in class.