222 §8.1

Our wrap up of lab 17 will take a while. Here's a key: 1 - §8.4, 2 - §8.1, 3 - §8.1, 4 - §8.2, 5 - §8.1 or §8.2, 6 - §8.4, 7 - §8.1. You probably notice lots of §8.1 here. Let's get started.

Remember where we begin. Before this chapter you have two possible integration methods: 1. the fundamental theorem of calculus which is important, but not much of a method - it says "what is this the derivative of?" and if you know the answer then you're done, otherwise it doesn't help. 2. substitution/compensation. I hope you remember and are ready to use substitution in its full capacity. Compensation is the simplified version. As an example, I would consider $\int \cos 2x dx$. Loosely the integral of cosine is sine, but the derivative of $\sin 2x$ is $2\cos 2x$, so to compensate we divide by the extra 2. Therefore $\int \cos 2x dx = \frac{1}{2}\sin 2x + C$.

Now for something new. Let's pick up with question 2 from lab 17. $\int xe^x dx$. I don't think we know what this is the derivative of, and there really aren't any viable substitutions ($u = x$ never ever helps, just changes letters, $u = e^x$ is the only possibly helpful one, and it doesn't do anything, and letting $u = xe^x$ would only help if we already knew the answer, so that's pretty much never helpful either). What we really want is how to integrate products. Remember differentiating products was the first surprise in derivatives. You don't differentiate a product by just taking the derivative of the parts and multiplying them, so you won't expect that to work for integrals either, and probably it's going to be worse. So, let's start by remembering the product rule, (1) what is $(fg)'$? This is a good opportunity to say now - get out of bad habits. Please do *not* change the order of f and g. There's no reason for it, and It will only get you into trouble in calc 3. Ok, now that you have the product rule, integrate each of the terms and recall that the integral and derivative undo each other. (2) What do you get? I hope $fg = \int f'g + \int fg'$. We may solve this for the leftmost term by moving the other to the right to get $\int fg' = fg - \int f'g$. This formula is sometimes called *integration by parts*. I call it the "backwards product rule", because that's what it is. I don't recommend memorising it (or much of anything else), but understanding where it comes from, since the steps above are easy to reproduce. Notice that it is somewhat disappointing - it takes one integral and trades it for another, we just hope the new one will be easier to work with. Now, back to our example ...

We want $\int xe^x dx$. Looking at our formula: $\int fg' = fg - \int f'g$, we see that we need to identify two "parts". Comparing the left and the right, notice that one of the parts we will be differentiating ($f \to f'$) and one we will be integrating ($g' \to g$), and yes to even use the formula we need to do a little bit of integration. We need to choose to make that not-so-bad, too. (3) To get going, what are the two parts? (For precision sake, be careful to include dx with g' and then with f'.) (4) Which one would you rather differentiate? [one of them doesn't matter, but the other one is made more complicated with differentiation]. (5) Complete the following:

$$f = \qquad\qquad\qquad\qquad g' =$$

$$f' = \qquad\qquad\qquad\qquad g =$$

$$\int fg' = fg - \int f'g =$$

So far you should have $\int xe^x dx = xe^x - \int e^x dx$. We've now completed our trade. I think the new integral is nicer, so let's just do that one by FTC to get a final answer of $\int xe^x dx = xe^x - e^x + C = (x-1)e^x + C$.

The next question in lab is $\int x^2 e^x dx$. Clearly this is similar to the last one. (6) What do we use as f? What as g'? The choices are probably similar. (7) Complete a table as above. Afterward you should see $\int x^2 e^x dx = xe^x - \int 2xe^x dx$. Note this is the same as $xe^x - 2\int xe^x dx$, and the integral is now identical to the first example. (8) Either use your work above or use the backwards product rule again. In either case, get $(x^2 - 2x + 1)e^x + C$.

What about question 3? Let's jump to $\int x^6 \ln x dx$. (9) What are the two parts? (10) Which part is which? You might be tempted to decide based on x^6, but remember last time we had freedom because e^x was the same either way. Now what about $\ln x$? If $g' = \ln x dx$, what is g? That would be a problem. So we only really have one choice here. (11) Complete another table. This should give you $\int x^6 \ln x dx = \frac{1}{7}x^7 \ln x - \int \frac{x^7}{7x} dx$. (12) Finish this integral.

Partway through I asked for the integral of $\ln x$. That seems worth knowing, and we don't yet. (13) Apply the above method to $\int \ln x dx$. There is one tricky question - (14) what are the two parts? (15) Once you get that the work is pretty quick. Your final answer should be $\int \ln x dx = x - x \ln x + C$.

We have one more left from lab: $\int \sin x \sin 2x dx$. This one is interesting in a new way. To get started, we make some choices. The only difference is the 2, but I guess since it gives us some suggestion for making our choices, we might as well use it. (16) Complete the first step to get $\int \sin x \sin 2x dx = -\cos x \sin 2x + \int 2 \cos x \cos 2x dx$. Continuing again may seem discouraging, but persevere. Be careful, if we choose to integrate $\cos 2x$ we'll be back where we started. Which will be correct, but not helpful. The other choice leads to curious results. (17) Complete another round of integration by parts. You should now have

$$\int \sin x \sin 2x dx = -\cos x \sin 2x + 2 \sin x \cos 2x + \int 4 \sin x \sin 2x dx.$$

Now, at this point you have reasons to be discouraged. Our goal in integration by parts is to trade one integral for an easier one. In the first step we switched sines to cosines. And, now, the integral is *the same* as what we started with. How does that hep? The surprising thing is that *it does*, and we're done with all the hard work. It is *good news* that we have the same integral on both sides. Now, (18) subtract the integral from the right to the left. (19) Combine them into one integral, pull the constant out of the integral and divide it to the left. (20) Don't forget $+C$ and you should have your final answer:

$$\int \sin x \sin 2x dx = \frac{1}{3}(\cos x \sin 2x - 2 \sin x \cos 2x) + C.$$

And, somehow we traded the integral for itself and made progress in the process.

Lots to ask about for class. I think that's plenty for here today.