222 §8.4

To get us started today, (1) please find a common denominator and write this as one fraction:

$$\frac{3}{x}+\frac{5}{x^2}+\frac{1+2x}{x^2+4}+\frac{4}{x-2}$$

That exercise should sound familiar. It should be the kind of thing you did in high school classes before calculus. A reasonable answer would be

$$\frac{9x^4-4x^3+16x^2-4x-40}{x^2(x-2)(x^2+4)}$$

The most important question ... was this helpful? (2) Which would you rather integrate, the original or the combined form? That answer (which should be obvious) is the point of today's work. To review our calculus work, (3) integrate (hint: separate the third terms into two, and remember that we did ones like the first one in §7.6, in fact, secretly, it's the main reason we talked about inverse trigonometry). You should get

$$3\ln x-\frac{5}{x}+\frac{1}{2}\tan^{-1}(\frac{x}{2})+\ln(x^2+4)+4\ln(x-2)+C$$

That's some background. Really nothing new yet. One more simple thing, as we're getting started. What about $\int\frac{x^3+2}{x-4}dx$? Again, it would be better if we could break it up. How? Long division is the key here. (4) Long divide and then integrate. You should get $\frac{1}{3}x^3+2x^2+16x+66\ln(x-4)+C$.

The real idea here concerns the first example. The important point is that having a common denominator is now inconvenient. So, the idea is to find a way to un-common denominator. Let's start with an example from lab 17 question 1: $\int\frac{1}{(x+2)(x-7)}dx$. Again, we need to think backwards. If this were the answer to a question like our opening question, what would we have started with? Probably two terms, one with $(x+2)$ as a denominator and one with $(x-7)$. What numerators? We don't know yet. So, we want to find A and B so that

$$\frac{1}{(x+2)(x-7)}=\frac{A}{x+2}+\frac{B}{x-7}$$

Doing this isn't calculus, but once we get it done, the calculus part will be easy. So, how to find A and B? (5) Start by clearing fractions. Then somehow solve for A and B. I have two ways, one which always works, and one which is usually easier. I'll talk about the way that always works first. Take the equation, and create new equations based on the coefficients: constants: $1=-7A+2B$ and linear: $0=A+B$. We now have two equations and two unknowns. (6) Solve them. (7) Substitute back in for A and B and integrate the original to get $\frac{1}{9}(\ln(x-7)-\ln(x+2))+C$.

Here's another way to find A and B. The equation produced after clearing fractions is true for all values of x. (8) What value of x is particularly well-suited for finding B? Do so. What value of x is well-suited for finding A? Do so. The values should produce the same answers as above.

That's the simplest case, but as you saw in the opening example, things can get more complicated. Consider $\int \frac{5x^2+18x-1}{(x+4)^2(x-3)}dx$. For this case, we have a potential for three parts:

$$\frac{5x^2+18x-1}{(x+4)^2(x-3)} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{x-3}$$

Notice, just like above, the first and second term could both be there. It might be that $A = 0$, be we cannot assume it, and it needs to be there as a possibility. As the best indicator, think about our first example. Now, back to work. (9) Clear fractions. Using well-chosen values you can find B and C. If you want to use this approach for A, do it after you've found B and C. Or work with coefficients. (10) Somehow find A, B, C:

$$\frac{3}{x+4} - \frac{1}{(x+4)^2} + \frac{2}{x-3}$$

(11) Integrate to get $3\ln(x+4) + \frac{1}{x+4} + 2\ln(x-3) + C$. Here is an example of how to begin an even more involved problem:

$$\frac{x^4-2x-1}{(x+8)^3(x-1)^2} = \frac{A}{x+8} + \frac{B}{(x+8)^2} + \frac{C}{(x+8)^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}.$$

One more variation and that will exhaust all of this idea. Consider $\int \frac{x-1}{x(x^2+1)}dx$. In this case we have a quadratic factor that can't be factored. Again this should remind you of something from the opening example. Like the pieces in that example, here we will need

$$\frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

So, (12) again clear fractions. Well-chosen values don't take us very far, but you can (13) find A. Probably going to be better to use coefficient equations. (14) Make sure you see how these equations arise: $A + B = 0, C = 1, A = -1$. From there it should be direct to solve for A, B, C. (15) Do so, and finish the integral to get

$$-\ln x + \frac{1}{2}\ln(1+x^2) + \tan^{-1} x$$

To wrap up - another example of how to begin a long problem:

$$\frac{27}{(x^2+x+1)^2(x-1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^2}$$

And the great news is that the methods discussed here are all you will ever need for integrating any rational expressions. Not to say that the work won't be difficult, never any new ideas. That's a nice accomplishment.