222 §10.10 We have reached the end of our long journey in Chapter 10. Congratulations, you have made it through the hardest part of the course. I promise you that the rest of the material will be easier to see. (I intentionally put the challenging part in the middle so that we could relax a little this time of year.) So, what is left? Some ways of using series.

Historically the most famous series is the binomial series. It is the series produced from the function $f(x) = (1+x)^n$. (1) Use Taylor series and take derivatives to find the pattern in order to find an infinite series for the function. You should be able to reproduce a familiar looking formula:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

At least I hope that is familiar looking. There are two catches to this. The first is that n doesn't need to be a natural number, it could be -1 or $\frac{1}{3}$. But, if it is, then we need to be a little more careful about $\binom{n}{k}$. The second catch is that for polynomials, this is a finite sum just up to degree n, but for other exponents this is an actual infinite series. In your work for question (1) I expect you took derivatives and then divided by $n!$ and somewhere found the coefficients, the "binomial coefficients" which we thus define as $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$, which I hope you see matches up with any definition you have had before for $\binom{n}{k}$. To see how this works (2) write the expansion for $(1+x)^6$, (3) write the first five terms of the series for $\frac{1}{1+x}$. Notice that you could also do this using a geometric series. (4) Write the first five terms of the series for $\sqrt[3]{1+x}$.

We began our work on series by discussing the benefits for integration. Although $\int \cos x^2 dx$ is impossible in closed form, finding a series for it is not too bad. (5) Please do so.

While derivatives usually don't cause much challenge, series can also be helpful with limits. Here's a simple one that is quite nice with series. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$. (6) Substitute a series for $\cos x$. (7) Simplify. (8) Take limits. Yes, I know you could have done it with Bernoulli's rule twice, but I think you might see that this way is easier. (9) Try the same again with $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$. One hint: you don't need a lot of terms for the $\tan x$ sequence. The answer is $-\frac{1}{2}$.

Finally, here's a curious backwards use. What is the hundredth derivative of e^{-x^2} at $x = 0$? You wouldn't want to compute them out to find this, but here's a way. There is a term of the series centred at zero that is $\frac{f^{(100)}(0)}{100!}x^{100}$. Knowing that and knowing what the series is, (10) find the desired derivative.

Tomorrow in class I will talk about the series for $e^x, \cos x, \sin x$. I hope you have noticed that they look similar. We will tie them all together, with a bow. (11) Please be prepared with them [always, and tomorrow in particular].