Before we get to new material, let's wrap up a the important parts of Lab 19. (1) Generalising one step further, what is $\lim_{n\to\infty} \frac{n^k}{a^n}$? (2) 3c from lab - what are the possible limiting behaviours of r^n, and for which r do they occur?

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Let's go back to the first time you worked with infinite series. It was probably long before you're thinking. (3) What is $\frac{1}{3}$ as a decimal? (4) What does that mean?

$$\frac{1}{3} = 0.\overline{3} = 0.3333333\cdots = 3\times 10^{-1} + 3\times 10^{-2} + 3\times 10^{-3} + \cdots + 3\times 10^{-n} + \cdots$$

Still, that's a bit much to take in. How do we deal with it? Step by step. Consider:

$s_1 = 0.3$

$s_2 = 0.33$

$s_3 = 0.333$

$s_n = 0.3333\cdots 3$ (with n 3s) $= \sum\limits_{k=1}^{n} 3\times 10^{-k}$

And so

$$\frac{1}{3} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\sum_{k=1}^{n} 3\times 10^{-k} = \sum_{k=1}^{\infty} 3\times 10^{-k}$$

And, if this sequence converges this is what we mean by $\frac{1}{3}$. Who knew? We'll consider in Lab 20 whether this series converges or not.

Let's consider an even simpler series. What about $s_n = \sum_{k=1}^{n} 7$? (5) What is s_n in closed form (without a summation symbol or dots, but of course including n)? (6) Now that you have a simple form of s_n, what is $\lim\limits_{n\to\infty} s_n$? (7) What if we replace 7 with 0.7? (8) What about 0.0000007?

Think about $\sum\limits_{k=1}^{n} \frac{k+1}{k}$. (9) What can you say about the terms (the summands that are being added)? Complete the following important statement

$$\text{If } \lim_{k\to\infty} a_k \neq \qquad\qquad\qquad\qquad , \text{ then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

Let's look at a couple more interesting examples what about $s_n = \sum\limits_{k=1}^{n} \frac{1}{k(k+1)}$? (10) What can you say about $\lim\limits_{k\to\infty} \frac{1}{k(k+1)}$? (11) What does the "important statement" say about $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)}$? The important answer to question 11 is *NOTHING*. So, what can we do? (12) Write out a few terms to get started:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \qquad\qquad + \qquad\qquad + \qquad\qquad + \cdots + \qquad\qquad +$$

[Notice: Write the first few terms, and then the penultimate and last terms.] Here's a helpful tip - remember partial fractions? Well, it happens that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. That is useful. Now, (13) complete the following:

$s_1 = \frac{1}{1 \cdot 2} = \frac{1}{1} - \frac{1}{2}$
$s_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} =$
$s_3 =$
$s_n =$

As a consequence of the above, we can now finish: $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} s_n$, and (14) what does this produce? Ok, we've now seen one example of a series (the name for the sum) converging to a particular value when the terms converge to zero.

Our next example is **THE MOST IMPORTANT EXAMPLE OF THE COURSE** So, pay attention. Consider the simple-looking series: $s_n = \sum\limits_{k=1}^{n} \frac{1}{k}$. Because it is so important, it has a name, the harmonic series. (15) What can you say about $\lim\limits_{k\to\infty} \frac{1}{k}$? Please remember that from this we know nothing about $s_n = \sum\limits_{k=1}^{n} \frac{1}{k}$. It could converge, and it could diverge. Because this is so important, we will see three different arguments for this. So, watch out for more. Here's the first: Suppose $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ did converge to a number, call it S. Then notice the following:

$$\sum_{k=1}^{\infty} = S = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right)$$

(16) Why is this so? Explain carefully - it's not sophisticated, but make sure you understand. (17) What do you get when you add the terms in parentheses? (18) What variable represents the answer? (19) Why is this a problem? (20) Why does this mean that **THE HARMONIC SERIES DIVERGES**?

Let's wrap up by returning to our important statement

$$\text{If } \lim_{k\to\infty} a_k \neq 0, \text{ then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

we now see that the converse is *not true*, i.e that

$$\text{If } \lim_{k\to\infty} a_k = 0, \text{ then } \sum_{k=1}^{\infty} a_k \text{ may converge } \textit{or} \text{ diverge.}$$

To make this clear once more, because it just matters that much: if $\lim\limits_{k\to\infty} a_k \neq 0$, then we're done, and it diverges and nothing more needs to be said. If $\lim\limits_{k\to\infty} a_k = 0$ then more work needs to be done. Similar to the $\frac{0}{0}$ case. And this is what makes series interesting, and why we need to be careful.