Before we get to new material, let's wrap up the important parts of Lab 20. First consider geometric series $\sum_{k=0}^{\infty} ar^k$. To be clear, write out the first 4 terms of this series. (1) What is the first term? (2) How do you get from one term to the next? Like we have before, please consider $s_n = \sum_{k=0}^{n} ar^k$. (2) Write out the first four and the last two terms of this finite series (use dots between). (3) Compute rs_n. (4) Compute $s_n - rs_n$. (5) Solve for s_n, and take the limit as n goes to infinity. (6) How do your results compare with your experimental results in lab for question 1e.? Geometric series are one of our most important series. Depending on how you count, our second or third most important example. Please make sure you have this information ready.

222 §10.3 The next examples in lab were the p-series: $\sum_{k=0}^{\infty} \frac{1}{k^p}$. In lab you had some experiments to deal with them. Here we have a new method to verify for certain. (7) Draw a picture of a function, $f(x)$, in the first quadrant that is continuous, positive, and decreasing (use $f(x) = \frac{1}{x^2}$ if you like, it's a nice example and will be useful when we get back to the story). The method we will use applies *only* to functions that are *continuous, positive, and decreasing.* Consider the area computed by $\int_1^{\infty} f(x)dx$. (8) Draw in rectangles for both the left-hand and right-hand approximations (using a base of 1 for each rectangle). You now have a very important picture. It has three quantities - the integral area, and each of the two rectangle areas. (9) Which is largest? (10) What is smallest? (11) What sum gives the area for each of the regions? After all of the above, check to see that you now have:

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n)$$

Notice the left and right approximations, as we have seen before, only differ at their end-points, and since they end at infinity, they only differ at their beginning. (12) Please complete this equation, which should be apparent by writing out a few terms: $? + \sum_{n=2}^{\infty} f(n) = \sum_{n=1}^{\infty} f(n)$.

(13) Now, solve for $\sum_{n=2}^{\infty} f(n)$ in terms of $\sum_{n=1}^{\infty} f(n)$and substitute it into the above inequality on the left. This gives:

$$\sum_{n=1}^{\infty} f(n) - f(1) \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n)$$

That's all set-up. Here's the key mindshift. Instead of thinking of this picture and inequality as a way to approximate the integral, we can think of it as a way to approximate the sum. Think about this - it's the same work, but used in the opposite way. (14) To do so, solve each of the two inequalities for the sum in terms of the integral and then reassemble the inequality. This should produce:

$$\int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^{\infty} f(x)dx$$

And notice, suddenly, we have a way to approximate series, as long as we follow the rules we started with (continuous, positive and decreasing) and we can integrate it. Now, those are a long list of rules, but there are many examples for which this is quite helpful.

Let's go back and apply this to our lab examples. (15) What does this say about the case when $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$? It should completely answer the question. And we should notice something ... if the integral diverges (as it does here), the series must diverge as well. (16) What does it say about $\sum_{n=1}^{\infty} \frac{1}{n^2}$? I hope you find that it converges to some number between 1 and 2. And then also notice that this isn't particularly accurate. Can we get more accurate? Yes, we can. In lab you might have computed that $\sum_{n=1}^{100} \frac{1}{n^2} = 1.63498390018$. We can now combine this with the integral test for $\sum_{n=101}^{\infty} \frac{1}{n^2}$ to get a nice approximation.

$$\int_{101}^{\infty} \frac{1}{x^2} dx \leq \sum_{n=101}^{\infty} \frac{1}{n^2} \leq \frac{1}{101^2} + \int_{101}^{\infty} \frac{1}{x^2} dx$$

(17) Compute the integral to find

$$\frac{1}{101} \leq \sum_{n=101}^{\infty} \frac{1}{n^2} \leq \frac{1}{101^2} + \frac{1}{101}$$

Putting this all together says that

$$1.63498390018 + \frac{1}{101} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.63498390018 + \frac{1}{101^2} + \frac{1}{101}$$

Arithmetic produces $1.64488489028 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.64498291989$, and that's pretty accurate. If you want even more accuracy, you could take the middle, to see that $\sum_{n=1}^{\infty} \simeq 1.64493390509$.

One more important thing to settle while we're here with the integral test. (18) For which values of p does the integral $\int_1^{\infty} \frac{1}{x^p} dx$ converge? Because of the integral test bounds, the same is true about the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. In particular, please note that this provides our *third* proof that the harmonic series diverges. Yes, it is that important that we have proven it three times.