222 §10.7 It's been a while, but recall the original motivation in Lab 22, and somewhat ironically, in §10.8. We were looking for polynomials to represent functions. And, at least with e^x we found that polynomials weren't quite enough. We needed infinite polynomials. Technically polynomials can't *be* infinite. These are not quite polynomials. They are *power series*, just like polynomials but infinite. Some reminders: (1) What is a power series for e^x? (It turns out there is more than one. This might not surprise you if you think back to Lab 22 and imagine restarting the question but centred at $x = 1$ instead of $x = 0$.) (2) What is a power series for $\sin x$? (3) What is a power series for $\cos x$? Become familiar with those three. They are important and not worth rederiving every time you see them.

In the opposite direction, (4) for what values of x does $\sum_{n=0}^{\infty} x^n$ converge? (5) To what function? Revisit the notes for §10.8 and see how we have begun to answer some of our questions and uncertainties from the beginning.

(6) What can you say about convergence of $\sum_{n=0}^{\infty} n!x^n$? Notice that the first three series each converge for all values of x. Thinking a little about convergence of power series in general, let's return to $\sum_{n=0}^{\infty} a_n x^n$. (7) What is one value of x for which this series is guaranteed to converge? This an important point – it says that the answer to "for which values does this series converge" is *never* "none". There's one other point to notice: if $\sum_{n=0}^{\infty} a_n x^n$ converges for one x value, then it must converge for smaller values as well. From this reasoning we notice that there are three options for the convergence of $\sum_{n=0}^{\infty} a_n x^n$:

Either it converges for all real values of x,

or there is a number R such that it converges for $|x| < R$ and diverges for $|x| > R$,

or it converges only for $x = 0$.

(8) For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge? This is a multistep process. First use the ratio test. From this you should find that it converges for the values on the interval $(-1, 1)$. What about the endpoints? They must be tested using other means. In this case, we see that for $x = 1$ we have the harmonic series which diverges, and for $x = -1$ we have the alternating harmonic series which converges. Therefore this series converges exactly on the interval $[-1, 1)$.

One final point that will show up in Lab 23: there's nothing special about $x = 0$, actually, just about the function being written as x^n. To get us going for class tomorrow, (9) what can you say about the convergence of $\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n n^2}$?

That was pretty tedious and technical. Let's end this with something a little more enjoyable. Finding derivatives isn't always the best way to find series. We have used this idea once with e^{-x^2}, which we didn't find by derivatives, but by substituting $-x^2$ into the series for e^x. (10) Do that again. (11) Find a series for $x^3 \cos x^2$ without taking any derivatives.

(12) What is the series for $\frac{1}{1-x}$? (13) How does that function relate to $\ln(1 - x)$? (14)

Use this to find a series for $\ln(1-x)$. (15) How does $\frac{1}{1-x}$ relate to $\frac{1}{(1-x)^2}$? (16) Use this to find a series for $\frac{1}{(1-x)^2}$.

(17) What substitution can you make into $\frac{1}{1-x}$ to get $\frac{1}{1+x^2}$? (18) Do this. (19) Where have you seen $\frac{1}{1+x^2}$ before? (20) Find a series for $\tan^{-1} x$. Wasn't that fun? It always feels to me like getting something for nothing. No taking long lists of derivatives and looking for patterns. I recommend looking for these nice ways to find series whenever you can.