In chapter 8 we learned, if nothing else, that integration is difficult, in fact, sometimes impossible. What if we could instead just integrate polynomials? They're great and easy to work with. It turns out that in most situations we can: in lab 22 we saw that we can use polynomials (or something like polynomials, but with infinitely many terms) in the place of functions. In that lab, we looked at finding a polynomial to use for $f(x) = e^x$. Notice: that's not particularly valuable, since we can so easily do calculus on e^x. But, once we have a polynomial for e^x, we have a polynomial for $f(-x^2) = e^{-x^2}$, which is both famously nonintegrable, but also exceptionally important as the standard normal curve.

After finding a function via experiment, a more systematic method is considered. Let's look at question 7. Let's try to find a polynomial that will work for a function $f(x)$. Our principle will be to match coefficients by matching derivatives. Suppose

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Let's start by substitution $x = 0$. (1) What does this produce? (2) Now differentiate both sides. (3) Substitute $x = 0$ again. On the left hand side we have $f'(0)$; (4) what is on the right hand side? Let's try it again. Differentiate and substitute in $x = 0$, and (5) what is $f''(0)$? And, continuing, (6) what is $f'''(0)$? And, now generalising, (7) what is $f^{(n)}(0)$ (this denotes the nth derivative? (8) Now, take one step further and invert each of these equations to solve for a_n. Returning to the above long equation, you should then have:

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{4!}x^5 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

That's pretty exciting, we now have a formula for taking any function and converting it into a (probably infinite) polynomial. How well does it work?

Well, sometimes great and sometimes less so. Consider two more examples of this, and we'll discuss more in class.

What about the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(9) Graph the function. (10) What is $f'(0)$? Technically you would need to do this with limits, but graphing $f'(x)$ and looking at zero should give you an idea. (11) What about $f''(0)$? (12) What does $f^{(n)}(0)$ look to be? (13) What does this say about the polynomial version of this function? (14) How does it work as an approximation?

(15) Apply the method of finding derivatives to find a series for $\sin x$. (16) Take the derivative of that series to find a series for $\cos x$. We will use these two series along with the one for e^x regularly. Please keep them available in your mind.

Let's look at our last example, which is better than the previous, but not as good as the first. Consider $f(x) = \frac{1}{1-x}$. Unlike the previous, all the derivatives are defined at $x = 0$. (17) What is $f(0)$? (18) $f'(0)$? (19) $f''(0)$? (20) $f'''(0)$? (21) Sensing a pattern? (22) Will it continue? (23) What does this say about the polynomial version of this function? Ok, that might be reasonable. (24) Check to see that it works for $x = 0$. (25) What about $x = 1$?

(26) Does the answer seem reasonable for $x = 1$? (27) What about $x = 2$? (28) What about $x = \frac{1}{2}$? There are some questions here, lots of them, in fact. The point of doing §10.8 this early is to show you why we care about functions as polynomials, but also that we can get some crazy answers if we're not careful. Chapter 10 will be all about learning to be more careful.