Wrap up on Lab 23: the most important thing about this lab is *seeing* what interval of convergence means. I believe this is a very powerful image and tells you what your computations are actually finding, and before I saw this lab, I never really knew what I was computing. The rest is a combination of practice and getting familiar with how the pictures look.

222 §10.9 Ok, we now have these Taylor series, and they converge in some places. Probably it's not too practical to think about using the whole infinite series, though. Probably we will just use a few terms from the beginning. But, if we do that, we would like to know how accurate our answer is. And, to get there we're going to revisit something you might not have thought carefully about in Calc I: the mean value theorem. This theorem says

There exists a real number c between x and a such that $f'(c) = \frac{f(x)-f(a)}{x-a}$.

To see the connection to Taylor polynomials, (1) solve the above equation for $f(x)$. Notice that what it is equals is very close to a Taylor polynomial of first degree with c replaced for a in the last term. For comparison, here is a higher degree version:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(c)}{3!}(x-a)^3$$

Notice that we've really done *two* things. The one that you probably notice more clearly is the introduction of c, some *unknown* value between a and x (and yes the unknown part is definitely potentially inconvenient), but much more importantly, we have changed an infinite *series* into a finite sum. We've made it something we can work with, except for that pesky problem of the unknown c.

Let's consider how this works in a particular example. What if we use a second-degree Taylor polynomial to approximate $e^{-0.1}$? (2) What is the polynomial result? (3) What do we get at $x = -0.1$? You should get 0.905. So, there's an approximation, but that's useless unless we know how close it is. This is where we look at the last term: $\frac{f'''(c)}{3!}(x-a)^3$. (4) What is that term in this case [still including a c]? (5) What two values is c between? In approximating error, we want to know how bad it can be. (6) What value of c in this range makes this term as large as possible (in absolute value)? (7) So, what is the maximal possible error? The last answer should be $\frac{1}{6000}$. So, $e^{-0.1}$ is within about 0.00017 of 0.905. How does your calculator feel about that? (8) What does it say $e^{-0.1}$ is? (9) And what is the difference between the approximation 0.905 and $e^{-0.1}$? (10) And how close it is to 0.00017? Looks pretty good to me.

One more example here . . . in a practical circumstance it would be more reasonable to have someone tell you how accurate they want the approximation to be rather than for you to tell them how accurate your approximation was. For an example of this, approximate $\cos 0.1$ to within 5 decimal places, (i.e. more accurate than 0.000005). The real question here is "how many terms?". The approach is to make the error term less than 0.000005. (11) What is the nth degree error term? Finding the right n is a little tricky, because we're not "solving" for it, but rather experimenting until we find it. Because of the fact that every other term is zero, we're looking for an even n such that $\frac{\cos c}{(n+2)!}(.1)^{n+2} < 0.000005$ for the right value of c. c is between 0 and 0.1. The value that makes $\cos c$ largest is $c = 0$. So, the question is when is $\frac{1}{(n+2)!}(.1)^{n+2} < 0.000005$? $n = 3$, makes the exponent 5, and is surely sufficient, producing $\frac{1}{12{,}000{,}000}$, about 0.000000083, but do we need to work this hard? What about $n = 2$? There we have $\frac{1}{(4)!}(.1)^4 = \frac{1}{240{,}000}$, about 0.00000417, which is just barely small enough. But wait, we said even integer anyway, so it's good we use 2, not 3. (12) So, what polynomial does that say to use? (13) And what approximation does that give? The answer to the second question should be 0.995. (14) How does it compare to the actual value?