## 223 Assignment 1 Solutions

§9.1.1 Suppose that the level curves of a function g(x, y) are horizontal lines. What does that imply about g?

If the level curves are horizontal lines, this says that all output values are the same for any x value, this is why the level curves are horizontal lines. Therefore this says that g(x, y) does not depend on x. It could be any function, but it is independent of x, so g(x, y) = g(y). For example  $g(x, y) = y - \sin y$ , or any other function only of y.

Suppose that the level curves of the function f are parallel straight lines. Must f be a plane? Justify your answer.

One answer is that the examples above are horizontal straight lines which are parallel, and they are not a plane, so there. The same thing can happen with vertical straight lines if f(x, y) = f(x) and is independent of y. But, this is the simplest situation. There are more interesting cases. I could be sneaky and give you an example without telling you how, but this is even better. One example is any function than can be written as f(y-x), e.g.  $f(y-x) = (y-x)^2 - \sin(y-x)$ . Notice that on the entire line y = x this produces f(0), and similar for other diagonal lines. This generalises to  $f(y - \frac{3}{5}x + 7)$  or f(y - mx - b) for any constants m, b. This question is so great that I'll give 5/4 points to those who make these big connections.

§9.1.2 Suppose  $(x, y, z) \neq (a, b, c)$ , in particular this means that one of the coordinates is not equal, suppose without loss of generality that  $x \neq a$ . Because of this  $(x - a)^2 > 0$ . Consider  $d(P,Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ . Since  $(y-b)^2 \geq 0$  and  $(z-c)^2 \geq 0$ , we thus have that  $(x-a)^2 + (y-b)^2 + (z-c)^2 > 0$  and hence d(P,Q) > 0 as desired.

b Because  $(v-w)^2 = (w-v)^2$ , we see that d(P,Q) = d(Q,P) by switching each of the coordinates. c By the midpoint formula  $M = (\frac{x+a}{2}, \frac{y+b}{2}, \frac{z+c}{2})$ .

$$d(P,M) = \sqrt{\left(x - \frac{x+a}{2}\right)^2 + \left(y - \frac{y+b}{2}\right)^2 + \left(z - \frac{z+c}{2}\right)^2} = \sqrt{\left(\frac{x-a}{2}\right)^2 + \left(\frac{y-b}{2}\right)^2 + \left(\frac{z-c}{2}\right)^2}$$
$$= \sqrt{\frac{(x-a)^2}{4} + \frac{(y-b)^2}{4} + \frac{(z-c)^2}{4}} = \frac{1}{2}\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = d(P,Q)/2$$
$$d(M,Q) = \sqrt{\left(\frac{x+a}{2} - a\right)^2 + \left(\frac{y+b}{2} - b\right)^2 + \left(\frac{z+c}{2} - c\right)^2} = \sqrt{\left(\frac{a-x}{2}\right)^2 + \left(\frac{b-y}{2}\right)^2 + \left(\frac{c-z}{2}\right)^2}$$

9.2 #1: Prove properties 2 and 6 for 2-dimensional vectors (on p. 25). You argument should look like those above the list of properties. Be explicit about where you use properties for real numbers along the way.

$$\begin{aligned} 2.(\mathbf{v} + \mathbf{u}) + \mathbf{w} &= (\langle v_1, \dots, v_n \rangle + \langle u_1, \dots, u_n \rangle) + \langle w_1, \dots, w_n \rangle \\ &= \langle v_1 + u_1, \dots, v_n + u_n \rangle + \langle w_1, \dots, w_n \rangle \\ &= \langle (v_1 + u_1) + w_1, \dots (v_n + u_n) + w_n \rangle \\ &= \text{use associative for real numbers } \langle v_1 + (u_1 + w_1), \dots v_n + (u_n + w_n) \rangle \\ &= \langle v_1, \dots, v_n \rangle + \langle u_1 + w_1, \dots, u_n + w_n \rangle \\ &= \langle v_1, \dots, v_n \rangle + (\langle u_1, \dots, u_n \rangle + \langle w_1, \dots, w_n \rangle) \\ &= \mathbf{v} + (\mathbf{u} + \mathbf{w}) \end{aligned}$$

 $6.a(\mathbf{v} + \mathbf{u}) = a(\langle v_1, \dots, v_n \rangle + \langle u_1, \dots, u_n \rangle)$ 

$$= a < v_1 + u_1, \dots, v_n + u_n > = < a(v_1 + u_1), \dots, a(v_n + u_n) >$$

= use distributive for real numbers  $\langle av_1 + au_1, \ldots, av_n + au_n \rangle$ 

$$= \langle av_1, \dots, av_n \rangle + \langle au_1, \dots, au_n \rangle$$
$$= a \langle v_1, \dots, v_n \rangle + a \langle u_1, \dots, u_n \rangle$$

 $= a\mathbf{v} + \mathbf{a}\mathbf{u}$ 

 $\S9.2 \#15$  from textbook.

(a)  $|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5.\mathbf{u} = <\frac{3}{5}, \frac{4}{5} > |\mathbf{u}| = 1$ . The direction of  $\mathbf{u}$  is the same as  $\mathbf{v}$ .

(b) 
$$\mathbf{w} = \langle 3, -3 \rangle, |\mathbf{w}| = 3\sqrt{2}, \mathbf{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$$
 is a unit vector in the direction of  $\mathbf{w}$ .

(c)  $|\mathbf{v}| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$ .  $\mathbf{u} = \langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \rangle$ .  $|\mathbf{u}| = 1$ . The direction of  $\mathbf{u}$  is the same as  $\mathbf{v}$ . (d)  $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$  is a unit vector in the direction of  $\mathbf{v}$ .  $-\frac{\mathbf{v}}{|\mathbf{v}|}$  is also parallel to  $\mathbf{v}$ , but is in the opposite direction.

§9.3.1 a. This uses  $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos \theta$ . So we compute:  $\mathbf{v} \cdot \mathbf{w} = 2 + 6 - 12 = -4$ ,  $|\mathbf{v}| |\mathbf{w}| = 1$  $\sqrt{(1+4+9)(4+9+16)} = \sqrt{406}$ . Solving for  $\cos \theta = \frac{-4}{\sqrt{406}}$ . Because this is negative,  $\theta$  is obtuse.

b. Our formula for projection of  $\mathbf{v}$  onto  $\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w} = \frac{-4}{29} \mathbf{w} = \left(\frac{-8}{29}, \frac{-12}{29}, \frac{16}{29}\right)$ . c. We have  $0 = \mathbf{x} \cdot \mathbf{v} = x_1 + 2x_2 + 3x_3$  and  $0 = \mathbf{x} \cdot \mathbf{w} = 2x_1 + 3x_2 - 4x_3$ . Use the first equation to eliminate  $x_1$  in the second, obtaining  $-x_2 - 10x_3 = 0$ . And use this to eliminate  $x_2$  in the first, obtaining  $x_1 - 17x_3 = 0$ . Now we choose a value for  $x_3 = 1$ . This produces the vector (17, -10, 1). Check that both dot products do give zero.

#12 from textbook. (a) To find the angle at P, we must make P like the origin, so we subtract P.  $Q - P = \langle -2, -4, 5 \rangle$  and  $R - P = \langle 1, 2, 1 \rangle$ . Now compute the angle between these two vectors.  $\cos \theta = \frac{-2-8+5}{\sqrt{45}\sqrt{6}}$ , so  $\theta \simeq 107.72^{\circ}$ .

Similarly, to find the angle at  $Q, P - Q = \langle 2, 4, -5 \rangle, R - Q = \langle 3, 6, -4 \rangle$ . So at  $Q, \cos \theta = \frac{6+24+20}{\sqrt{45}\sqrt{61}}$ so  $\theta \simeq 17.38^{\circ}$ .

Finally, at R, P-R = <-1, -2, -1 >, Q-R = <-3, -6, 4 >. And at R,  $\cos \theta = \frac{3+12-4}{\sqrt{6}\sqrt{61}}$ , so  $\theta \simeq 54.90^{\circ}$ . I am relieved to see the sum of these angles.

(b) I'll let  $\mathbf{a} = P - Q < 2, 4, -5 > \text{and } \mathbf{b} = R - Q = < 3, 6, -4 > \mathbf{proj_ba} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} = \frac{50}{61} \mathbf{b} = < \frac{150}{61}, \frac{300}{61}, \frac{-200}{61} > \mathbf{proj_{\perp b}a} = \mathbf{a} - < \frac{150}{61}, \frac{300}{61}, \frac{-200}{61} > = < \frac{-28}{61}, \frac{-56}{61}, \frac{-105}{61} > \mathbf{proj_{\perp b}a}$  is perpendicular to QR through P, so it is surely a height of the triangle, with base QR. So, the area is half of the product of these two lengths:  $\frac{1}{2} \frac{7}{61} \sqrt{305} \sqrt{61} = \frac{7}{2} \sqrt{5} \simeq 7.826$  square units.

§9.4.1 (a) Explain how the cross product can be used to determine whether three points in space are collinear.

Suppose the three points are  $P_1, P_2$ , and  $P_3$ . If they are collinear, then their differences are parallel, so compute  $(P_1 - P_2) \times (P_1 - P_3)$ , if this is **0**, then the three points are collinear.

(b) Describe a method for determining whether four points lie in the same plane.

This takes more visualisation than the first part. We have four points now,  $P_1, P_2, P_3$  and  $P_4$ . If they are all in the same plane, then their difference vectors will all lie in the same plane, like in (a). How do we tell if vectors all lie in the same plane? Their cross products will all be parallel (because they are all perpendicular to this one plane). So, if they all lie in the same plane then  $((P_1 - P_2) \times (P_3 - P_4)) \times ((P_2 - P_3) \times (P_4 - P_1)) = \mathbf{0}$ , and similarly for other pairings.

§9.4.2 Suppose v and w are two different nonzero vectors. If  $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ , what must be true about  $\mathbf{u}$ ?

There are many strategies we would have to a similar question ab = ac. The one that works best here would be ab - ac = 0 so a(b - c) = 0. In our case  $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$  so  $\mathbf{u} \times \mathbf{v} - \mathbf{u} \times \mathbf{w} = \mathbf{0}$ . By bilinearity we have  $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$  This tells us that **u** is parallel to  $\mathbf{v} - \mathbf{w}$ , which seems like some good information.