## 223 Assignment 2 Solutions

9.5 #1 Clarifying all roles here should clean this up.  $\mathbf{P}_0$  and  $\mathbf{X}$  are both vectors pointing to points on the plane  $\Pi$  (one might even say they are points on the plane  $\Pi$ ).  $\mathbf{P}_1$  is a point in space (I guess it could be on the plane  $\Pi$ , but it's probably natural to think of it as not being so). **n** is a *unit* normal to the plane  $\Pi$ . For the equation  $\mathbf{n} \cdot (\mathbf{X} - \mathbf{P}_0) = 0$ , note that  $\mathbf{X} - \mathbf{P}_0$  is a vector that points from  $\mathbf{P}_0$  to  $\mathbf{X}$ , and therefore is in the plane  $\Pi$ . Given this, and that we can base vectors where we want, since we're taking a dot product of the two vectors, it's probably good to think of n as based at  $\mathbf{P}_0$  as  $\mathbf{X} - \mathbf{P}_0$  is.  $\mathbf{P}_1 - \mathbf{P}_0$  is also based at  $\mathbf{P}_0$ . Draw this picture. Then draw the distance of  $\mathbf{P}_1$  from the plane  $\Pi$ . Notice that we get the same thing by projecting  $\mathbf{P}_1 - \mathbf{P}_0$  onto **n**. Because **n** is a unit vector the projection of  $\mathbf{P}_1 - \mathbf{P}_0$  onto **n** is simply  $(\mathbf{P}_1 - \mathbf{P}_0) \cdot \mathbf{n}$ . Notice that the sign of this projection could be negative if  $\mathbf{P}_1$  is on the opposite side of the plane from the direction of the chosen normal vector  $\mathbf{n}$ . To accommodate this we take absolute value of the dot product (not norm, remember the dot product is a scalar) this produces:  $|(\mathbf{P}_1 - \mathbf{P}_0) \cdot \mathbf{n}|$  or equivalently  $|\mathbf{n} \cdot \mathbf{P}_1 - \mathbf{n} \cdot \mathbf{P}_0|$ , as desired.

#2  $\ell_1$  is given by  $\mathbf{x} = (1, 1, 2) + t(3, -1, 4)$ . For comparison sake, as symmetric scalar equations this is  $\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z-1}{4}$ . For  $\ell_2$  we have  $\mathbf{y} = (1, 0, 3) + t(6, -2, 8)$  or  $\frac{x-1}{6} = \frac{y}{-2} = \frac{z-3}{8}$ . a. The lines are parallel since one direction vector (6, -2, 8) is two times the other, (3, -1, 4).

b. At the moment this seems to have little to do with the lines.  $\overrightarrow{PQ}$  is given by P-Q = (0, 1, -1). From this  $|\overline{PQ}| = \sqrt{2}$ .

c.  $\mathbf{v} = (3, -1, 4)$  is the direction of  $\ell_1$ . It's not a unit vector. It's length is  $\sqrt{26}$ . The scalar projection is then  $\overrightarrow{PQ} \cdot \frac{\mathbf{v}}{|v|} = (0, 1, -1) \cdot \frac{1}{\sqrt{26}} (3, -1, 4) = \frac{-5}{\sqrt{26}}$ . d. Now we're back to the two lines. Draw a picture of this situation. Make P and Q not connected by a

segment perpendicular to the lines. From this you should see a right triangle with hypotenuse  $\overrightarrow{PQ}$  and one leg as the scalar projection found in c. Therefore if d is the distance between  $\ell_1$  and  $ell_2$ , then  $d^2 + \frac{25}{26} = 2$ . So,  $d^2 = \frac{27}{26}$  and  $d = \sqrt{\frac{27}{26}}$ .

§9.6 #1 The spacecraft is at  $\mathbf{p} = (6, 5, -1)$  at t = 3. The velocity is  $\mathbf{r}' = (2t - 1, 1, 3/t^2)$  in particular at t = 3 this gives  $\mathbf{v} = (5, 1, 1/3)$ . Since the engines cut off, the traveling is only based on the velocity at that point, so it continues to travel along that velocity and the astronaut two hours later is at  $\mathbf{p} + 2\mathbf{v} =$ (16, 7, -1/3).

#12 a. Here are several views on this topic. Intuitively  $\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$  is  $\langle \cos t, \sin t \rangle$ but scaled in the two directions, by a horizontally and by b vertically. This meets the description of the ellipse, and there is surely one point in each direction, in particular the range of t:  $0 \le t \le 2\pi$  includes all directions. The only remaining question is if it satisfies  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , the given equation for an ellipse. To check we substitute  $x = a \cos t$  and  $y = b \sin t$  into the left side of the equation. We find  $\frac{(a \cos t)^2}{a^2} + \frac{(b \sin t)^2}{b^2} = \frac{(b \sin t)^2}{a^2}$  $\frac{a^{2}\cos^{2} t}{a^{2}} + \frac{b^{2}\sin^{2} t}{b^{2}} = \cos^{2} t + \sin^{2} t = 1, \text{ as desired.}$ 

b. Happily our parametrisation traverses counterclockwise already - it is based on the standard one for the unit circle, which proceeds counterclockwise starting at (1,0) and heading around the circle toward (0,1). So, the remaining parts are quite direct. a = 2 and b = 4, so use  $\mathbf{r}(t) = \langle 2 \cos t, 4 \sin t \rangle$ .

c. Much like in an equation for a circle, the constants shift the centre of the ellipse. This is a translation, in this case by vector  $\langle -3, 2 \rangle$ , so we use  $\mathbf{r}(t) = \langle 2\cos t - 3, 4\sin t + 2 \rangle$ .

d. This is the same as the previous, only backwards. What do we have? A translation by  $\langle 3, 1 \rangle$ , and scaling by 4 horizontally and by 3 vertically. The only lingering part is - what about the (2t)? That says traverse it twice for  $0 \le t \le 2\pi$  and go twice as fast. That is potentially important in some settings, but doesn't change the cartesian coordinate equation at all. Putting this together we have  $\frac{(x-3)^2}{4^2} + \frac{(x-1)^2}{3^2} = 1$ . Not so bad.

9.7 #1 Consider **u** and **v** as our coordinate axes. Note that we can basically integrate coordinate wise to find  $\mathbf{P}(t) = \mathbf{u} + t^2/2\mathbf{v}$  (check by differentiating component-wise). This produces a ray starting at  $\mathbf{u}$  in the direction of  $\mathbf{v}$ . Notice that all points are of the form  $\mathbf{u} + s\mathbf{v}$  for some s. This is the same line as  $\mathbf{u} + s\mathbf{v}$ , but traversed at a non-constant speed. Furthermore notice that since  $t^2/2$  is never negative, we only get one direction of the line, hence a ray.

#2 a. Probably plot this using maple. Ask me if you have questions. You should find a helix which starts at (1, 0, 0) and wraps around twice while moving up to  $z = 4\pi$ .

b.  $\mathbf{p}'(t) = (-\sin t, \cos t, 1), \mathbf{p}'(\pi) = (0, -1, 1), \text{ and } \mathbf{p}(\pi) = (-1, 0, \pi).$  So we need a line thru  $\mathbf{p}(\pi)$  in the  $\mathbf{p}'(\pi)$  direction:  $(-1, 0, \pi) + t(0, -1, 1).$ 

c. Again the plot could be produced via maple.

d.  $|\mathbf{p}'(t)| = |(-\sin t, \cos t, 1)| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$ . Arclength is the integral of speed over time. Because the speed is constant, arclength is merely the speed times the time, i.e.  $4\sqrt{2\pi}$ .

§9.8 #1 We need parametrisations for each of these. Once we get there, we should be fine. The first one is  $\mathbf{f}(t) = (5\cos t, 5\sin t, \frac{4}{6\pi}t)$  (in  $6\pi$  it makes 3 complete turns and by that time will go up 4) for  $0 \le t \le 6\pi$ . Similarly the second one is  $\mathbf{s}(s) = (3\cos t, 3\sin t, \frac{4}{10\pi}t)$  for  $0 \le t \le 10\pi$ . We follow the path to compute arclength for the first:  $\mathbf{f}'(t) = (-5\sin t, 5\cos t, \frac{4}{6\pi}), |\mathbf{f}'(t)| = \sqrt{5^2 + \frac{16}{36\pi^2}}$ . Since this is over a total time of  $6\pi$  we get an arclength of  $6\pi\sqrt{5^2 + \frac{16}{36\pi^2}} \simeq 94.3326$ .

For the second,  $\mathbf{s}'(t) = (-3\sin t, 3\cos t, \frac{4}{10\pi}), |\mathbf{f}'(t)| = \sqrt{3^2 + \frac{16}{100\pi^2}}$ . Since this is over a total time of  $10\pi$  we get an arclength of  $10\pi\sqrt{3^2 + \frac{16}{100\pi^2}} \simeq 94.3326$ . Some algebra confirms that these are actually the same number, although they do look a little different in these forms. So, both are the same length. Hm.

#2 We can parametrise this curve as  $\mathbf{p}(t) = (t, t^3)$ . Let's see what we find:

$$\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{s^3}$$

 $\mathbf{p}'(t) = (1, 3t^2)$ , and  $\mathbf{p}''(t) = (0, 6t)$ . Therefore the speed is  $\sqrt{1+9t^4}$ . And our formula follows as

$$\kappa = \frac{|1(6t) - (3t^2)(0))|}{\sqrt{1 + 9t^4}^3} = \frac{|6t|}{\sqrt{1 + 9t^4}^3}$$

At t = 1,  $\kappa = \frac{3}{5\sqrt{10}}$ 

That was the easy part. Now to find the circle. We want a circle with radius  $\frac{5\sqrt{10}}{3}$ . But there are plenty circles with that radius thru the point (1, 1). We want the one that is tangent with the tangent line at that point. The tangent line at that point has direction (1, 3). By dot product, we can see that a perpendicular direction is (-3, 1). Now we want to go a distance  $\frac{5\sqrt{10}}{3}$  in that direction. To be sure we get our distance correct, let's make our direction a unit vector:  $(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}})$  (conveniently we have  $\sqrt{10}$  showing up). So, we translate  $\frac{5\sqrt{10}}{3}(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}) = (-5, \frac{5}{3})$  from our tangent point of (1, 1). This gives our centre to be  $(-4, \frac{8}{3})$  and recall our radius is still  $\frac{5\sqrt{10}}{3}$ . So, finally a parametrisation of the circle is  $(-4 + \frac{5\sqrt{10}}{3}\cos\theta, \frac{8}{3} + \frac{5\sqrt{10}}{3}\sin\theta)$ . Some tricky work for something that seemed simple.

That is a way. Here is the more systematic way presented in class ...  $\mathbf{p}'(t) = (1, 3t^2)$ . from there  $\mathbf{T}(t) = \left(\frac{1}{\sqrt{9t^4+1}}, \frac{3t^2}{\sqrt{9t^4+1}}\right)$ . Taking derivatives we find  $\mathbf{T}'(t) = \left(\frac{-18t^3}{\sqrt{9t^4+1^3}}, \frac{6x}{\sqrt{9t^4+1^3}}\right)$ . At t = 1 we get  $\left(\frac{-18}{\sqrt{10^3}}, \frac{6}{\sqrt{10^3}}\right)$ . And now we converge to the same place as above, where we have a perpendicular direction that is parallel to (-3, 1). So, the remaining part of the solution is the same.