## 223 Assignment 4 Solutions

10.5 \#1 Let $z=(x, y)$ be a function of the Cartesian coordinates $x$ and $y$. Show that if the variable substitutions $x=r \cos \phi$ and $y=r \sin \phi$ are used to express in polar coordinates, then

$$
\frac{\partial^{2} f}{\partial x^{2}}+\frac{\partial^{2} f}{\partial y^{2}}=\frac{\partial^{2} f}{\partial r^{2}}+\frac{1}{r} \frac{\partial f}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} f}{\partial \phi^{2}}
$$

We'll work from the right side - because there's nothing to compute on left. So, let's grab some chain rule information:

$$
\begin{aligned}
& \frac{\partial f}{\partial r}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial r}=\frac{\partial f}{\partial x} \cos \phi+\frac{\partial f}{\partial y} \sin \phi . \\
& \frac{\partial^{2} f}{\partial r^{2}}=\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x} \cos \phi+\frac{\partial f}{\partial y} \sin \phi\right) \text { note at this point: } \frac{\partial f}{\partial x} \text { is likely a function of both } x \text { and } y \text {. } \\
& =\left(\frac{\partial^{2} f}{\partial x^{2}} \frac{\partial x}{\partial r}+\frac{\partial^{2} f}{\partial y \partial x} \frac{\partial y}{\partial r}\right) \cos \phi+\left(\frac{\partial^{2} f}{\partial x \partial y} \frac{\partial x}{\partial r}+\frac{\partial^{2} f}{\partial y^{2}} \frac{\partial y}{\partial r}\right) \sin \phi \\
& =\frac{\partial^{2} f}{\partial x^{2}} \cos ^{2} \phi+\frac{\partial^{2} f}{\partial y \partial x} \sin \phi \cos \phi+\frac{\partial^{2} f}{\partial x \partial y} \cos \phi \sin \phi+\frac{\partial^{2} f}{\partial y^{2}} \sin ^{2} \phi \\
& \frac{\partial f}{\partial \phi}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial \phi}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial \phi}=-\frac{\partial f}{\partial x} r \sin \phi+\frac{\partial f}{\partial y} r \cos \phi . \\
& \frac{\partial^{2} f}{\partial \phi^{2}}=\frac{\partial}{\partial \phi}\left(-\frac{\partial f}{\partial x} r \sin \phi+\frac{\partial f}{\partial y} r \cos \phi\right) \text { also uses product rule here } \\
& =-\left(\frac{\partial^{2} f}{\partial x^{2}} \frac{\partial x}{\partial \phi}+\frac{\partial^{2} f}{\partial y \partial x} \frac{\partial y}{\partial \phi}\right) r \sin \phi-\frac{\partial f}{\partial x} r \cos \phi+\left(\frac{\partial^{2} f}{\partial x \partial y} \frac{\partial x}{\partial \phi}+\frac{\partial^{2} f}{\partial y^{2}} \frac{\partial y}{\partial \phi}\right) r \cos \phi-\frac{\partial f}{\partial y} r \sin \phi \\
& =\frac{\partial^{2} f}{\partial x^{2}} r^{2} \sin ^{2} \phi-\frac{\partial^{2} f}{\partial y \partial x} r^{2} \cos \phi \sin \phi-\frac{\partial f}{\partial x} r \cos \phi-\frac{\partial^{2} f}{\partial x \partial y} r^{2} \sin \phi \cos \phi+\frac{\partial^{2} f}{\partial y^{2}} r^{2} \cos ^{2} \phi-\frac{\partial f}{\partial y} r \sin \phi
\end{aligned}
$$

So, now assembling the right side:

$$
\begin{aligned}
\frac{\partial^{2} f}{\partial r^{2}}+\frac{1}{r} \frac{\partial f}{\partial r}+ & \frac{1}{r^{2}} \frac{\partial^{2} f}{\partial \phi^{2}}=\frac{\partial^{2} f}{\partial x^{2}} \cos ^{2} \phi+ \\
& \frac{\partial^{2} f}{\partial y \partial x} \sin \phi \cos \phi+\frac{\partial^{2} f}{\partial x \partial y} \cos \phi \sin \phi+\frac{\partial^{2} f}{\partial y^{2}} \sin ^{2} \phi \\
& +\frac{1}{r} \frac{\partial f}{\partial x} \cos \phi+\frac{1}{r} \frac{\partial f}{\partial y} \sin \phi
\end{aligned} \quad \begin{aligned}
+ & \frac{\partial^{2} f}{\partial x^{2}} \sin ^{2} \phi-\frac{\partial^{2} f}{\partial y \partial x} \cos \phi \sin \phi-\frac{1}{r} \frac{\partial f}{\partial x} \cos \phi-\frac{\partial^{2} f}{\partial x \partial y} \sin \phi \cos \phi+\frac{\partial^{2} f}{\partial y^{2}} \cos ^{2} \phi-\frac{1}{r} \frac{\partial f}{\partial y} \sin \phi
\end{aligned}
$$

four pairs cancel leaving
$=\frac{\partial^{2} f}{\partial x^{2}} \cos ^{2} \phi+\frac{\partial^{2} f}{\partial y^{2}} \sin ^{2} \phi+\frac{\partial^{2} f}{\partial x^{2}} \sin ^{2} \phi+\frac{\partial^{2} f}{\partial y^{2}} \cos ^{2} \phi=\frac{\partial^{2} f}{\partial x^{2}}+\frac{\partial^{2} f}{\partial y^{2}}$, as desired.
\#17 We have: $V=I R$ and $R=\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right)^{-1}$. All of these quantities change over time. To get started, $\frac{\partial V}{\partial t}=\frac{\partial V}{\partial I} \frac{\partial I}{\partial t}+\frac{\partial V}{\partial R} \frac{\partial R}{\partial t}$. Similarly $\frac{\partial R}{\partial t}=\frac{\partial R}{\partial R_{1}} \frac{\partial R_{1}}{\partial t}+\frac{\partial R}{\partial R_{2}} \frac{\partial R_{2}}{\partial t}$. Putting one in the other we have: $\frac{\partial V}{\partial t}=\frac{\partial V}{\partial I} \frac{\partial I}{\partial t}+\frac{\partial V}{\partial R}\left(\frac{\partial R}{\partial R_{1}} \frac{\partial R_{1}}{\partial t}+\frac{\partial R}{\partial R_{2}} \frac{\partial R_{2}}{\partial t}\right)$. . Now, compute those that we can to get:

$$
\frac{\partial V}{\partial t}=R \frac{\partial I}{\partial t}+I\left(\frac{R^{2}}{R_{1}^{2}} \frac{\partial R_{1}}{\partial t}+\frac{R^{2}}{R_{2}^{2}} \frac{\partial R_{2}}{\partial t}\right)
$$

(Notice that along the way here we had $\left.\frac{\partial R}{\partial R_{1}}=-1\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right)^{-2}\left(-\frac{1}{R_{1}^{2}}\right)=-R^{2}\left(-\frac{1}{R_{1}^{2}}\right)=\frac{R^{2}}{R_{1}^{2}}\right)$. And for b., we merely substitute. $I=3, \frac{\partial I}{\partial t}=0.1, R_{1}=2$. $\frac{\partial R_{1}}{\partial t}=-0.2, R_{2}=1, \frac{\partial R_{2}}{\partial t}=0.5$. Notice here $R=\frac{2}{3}$.

$$
\frac{\partial V}{\partial t}=\frac{2}{3}(0.1)+3\left(\frac{\left(\frac{2}{3}\right)^{2}}{2^{2}}(-0.2)+\frac{\left(\frac{2}{3}\right)^{2}}{1^{2}}(0.5)\right)=\frac{2}{3} \text { volts } / \mathrm{sec}
$$

10.6 \#1 Let $g(x, y)=x^{2}-3 x y+6$ and let $C$ be the level curve of $g$ that passes through the point $\left(x_{0}, y_{0}\right)$. (Feel free to let Maple give you a contour plot if you wish to see them.)
a. Show that if $g\left(x_{0}, y_{0}\right) \neq 6$, then $\left(x_{0}, y_{0}\right)$ is a point on the curve described by the equation $y=$ $\frac{x^{2}+6-g\left(x_{0}, y_{0}\right)}{3 x}$.

For these first two parts, suppose $g\left(x_{0}, y_{0}\right)=c$, just to simplify notation. Then we have that the curve $C$ is given by $x^{2}-3 x y+6=c$, which we can solve for $y$ to get $y=\frac{x^{2}+6-c}{3 x}=\frac{x^{2}+6-g\left(x_{0}, y_{0}\right)}{3 x}$, as desired.
b. Suppose that $g\left(x_{0}, y_{0}\right) \neq 6$. Find a vector that is tangent to $C$ at $\left(x_{0}, y_{0}\right)$. [Hint: Use implicit differentiation.]

I'll follow the hint $\ldots$ starting with $x^{2}-3 x y+6=c$, we can differentiate implicitly to get $2 x-3 y-3 x \frac{d y}{d x}=0$, solving yields $\frac{d y}{d x}=\frac{2 x-3 y}{3 x}$, at $\left(x_{0}, y_{0}\right)$ this is $\frac{2 x_{0}-3 y_{0}}{3 x_{0}}$. At this point, the tangent line is therefore $y-y_{0}=$ $\frac{2 x_{0}-3 y_{0}}{3 x_{0}}\left(x-x_{0}\right)$. Letting $x=t$, we get $\left(t, \frac{2 x_{0}-3 y_{0}}{3 x_{0}}\left(t-x_{0}\right)+y_{0}\right)$ which gives a tangent vector of $\left(1, \frac{2 x_{0}-3 y_{0}}{3 x_{0}}\right)$ or equivalently $\left(3 x_{0}, 2 x_{0}-3 y_{0}\right)$.
c. Suppose that $g\left(x_{0}, y_{0}\right)=6$ and $x_{0} \neq 0$. Show that $\left(x_{0}, y_{0}\right)$ is a point on the line $y=x / 3$.

We start with $x^{2}-3 x y+6=6$, which gives $y=\frac{x^{2}}{3 x}$ as long as $x \neq 0$. This produces $y=\frac{x}{3}$ at $\left(x_{0}, y_{0}\right)$, as desired. For future work, this is in the direction $\left(1, \frac{1}{3}\right)=(3,1)$.
d. Suppose that $g\left(x_{0}, y_{0}\right)=6$ and $x_{0}=0$. Show that $\left(x_{0}, y_{0}\right)$ is a point on the line $x=0$.

We start with $x^{2}-3 x y+6=6$ and with $x=0$ we get $6=6$, which is true but not helpful. It tells us $y$ can be anything. Since we already know $x=0$ we get the line $x=0$.
e. Use parts b-d to show that in all cases the gradient of $g\left(x_{0}, y_{0}\right)$ is perpendicular to $C$ at $\left(x_{0}, y_{0}\right)$.

So, what is the gradient? $\nabla g(x, y)=(2 x-3 y,-3 x)$. At $\left(x_{0}, y_{0}\right)$ this is $\left(2 x_{0}-3 y_{0},-3 x_{0}\right)$. For part a-b., how does it compare to the tangent vector $\left(3 x_{0}, 2 x_{0}-3 y_{0}\right)$ ? Please notice by zero dot product we see they are perpendicular.

For part c., how does it compare with $(3,1)$ ? Remember for this point we deduced that $y_{0}=\frac{x_{0}}{3}$, so the gradient simplifies to $\left(2 x_{0}-x_{0},-3 x_{0}\right)=\left(-x_{0}, 3 x_{0}\right)$. And so they are perpendicular again by dot product.

Finally, if $x_{0}$, the gradient is $\left(-3 y_{0}, 0\right)$ which is on the $x$-axis and hence perpendicular to $x=0$. So, the result that the gradient is perpendicular to level curves is verified in all cases for this particular function.
\#2 Suppose that $f(x, y)=x^{2} y$. In what direction(s) from the point $(1,2)$ is the rate of change 3 ?
Here $\nabla f(x, y)=\left(2 x y, x^{2}\right)$, so $\nabla f(1,2)=(4,1)$. Now we seek a unit vector $\mathbf{u}$ such that $(4,1) \cdot \mathbf{u}=3$. Let $\mathbf{u}=\left(u_{1}, u_{2}\right)$, so we have $u_{1}^{2}+u_{2}^{2}=1$ and $4 u_{1}+u_{2}=3$. Substituting and rearranging gives $17 u_{1}^{2}-24 u_{1}+8=0$. Quadratic formula produces $u_{1}=\frac{12 \pm 2 \sqrt{2}}{17}$ and hence the directions are $\left(\frac{12 \pm 2 \sqrt{2}}{17}, \frac{\mp 8 \sqrt{2}+3}{17}\right)$.
10.7 \#1 If a continuous function of one variable has at least two local maxima, then it must also have at least one local minimum (think about drawing this picture). The situation is different for functions of two variables. Show that $f(x, y)=\left(x^{2}-1\right)^{2}+y^{2}$ has exactly three critical points - two local minima and a saddle point.

So, we find $\nabla f(x, y)=\left(2\left(x^{2}-1\right) 2 x, 2 y\right)=\left(4 x^{3}-4 x, 2 y\right)$. This equals the zero vector giving us critical points at $(-1,0),(0,0)$, and $(1,0)$. So, there are exactly three. It only remains to test. So, we try the second derivative test. So, we grab $\frac{\partial^{2} f}{\partial x^{2}}=12 x^{2}-4, \frac{\partial^{2} f}{\partial x \partial y}=0$, and $\frac{\partial^{2} f}{\partial y^{2}}=2$. This yields $\left|\begin{array}{cc}12 x^{2}-4 & 0 \\ 0 & 2\end{array}\right|=24 x^{2}-8$. at our points $x=-1,0,1$ this produces $16,-8,16$, respectively. We always have $\frac{\partial^{2} f}{\partial y^{2}}=2$, so we have minima at $x= \pm 1$ and a saddle point for $x=0$.

Pretty easy question - the point here is the different geometry that is possible
Choose one of $\# 5, \# 6, \# 7$ in the textbook. I guess this means I need to do all of them.
\#21 a. Draw a picture. Ask me if you want to see one.
b. $V=x y z$ using $x+2 y+3 z=6$ We can solve for $z=2-\frac{1}{3} x-\frac{2}{3} y$ and then substitute to find $V=2 x y-\frac{1}{3} x^{2} y-\frac{2}{3} x y^{2}$. (why didn't we want $V$ in terms of $y$ and $z$ ?)
c. $\nabla V=\left(2 y-\frac{2}{3} x y-\frac{2}{3} y^{2}, 2 x-\frac{1}{3} x^{2}-\frac{4}{3} x y\right)=\left(y\left(2-\frac{2}{3} x-\frac{2}{3} y\right), x\left(2-\frac{1}{3} x-\frac{4}{3} y\right)\right)$. One point is $(0,0,2)$, which surely must be the minimum. Each of the three points where one of $x$ or $y$ is zero is a minimum, i.e. $(6,0,0),(0,3,0)$, on the axes, but there is an interesting point $\left(2,1, \frac{2}{3}\right)$.
d. At each of the points including zero, the volume is zero. At the other point the volume is $\frac{4}{3}$. I really don't think this is worthy of more work. It's clear the maximum needs to occur off the axes, and that there will only be one such.
e. Clearly our point $\left(2,1, \frac{2}{3}\right)$ doesn't satisfy $\frac{1}{2} \leq x \leq 1$. Because this doesn't work our only option is find a maximum/minimum on the boundary. So, we set out. There are four sides to the rectangle. First $x=\frac{1}{2}$, where $V=y-\frac{1}{12} y-\frac{1}{3} y^{2}$. Similarly we get $V=2 y-\frac{1}{3} y-\frac{2}{3} y^{2}, V=2 x-\frac{1}{3} x^{2}-\frac{2}{3} x$ and $V=4 x-\frac{2}{3} x^{2}-\frac{8}{3} x$. It is tedious but routine to apply calc I techniques to teach of these to find the bounded minimum. Aside from corner points, which we need to check all of, we have these $\left(\frac{1}{2}, \frac{11}{8}\right),\left(1, \frac{5}{4}\right),(2,1),(1,2)$. The last is a corner point, and the penultimate is out of range. So, we check $\left(\frac{1}{2}, \frac{11}{8}\right),\left(1, \frac{5}{4}\right),(1,2),\left(\frac{1}{2}, 1\right),\left(\frac{1}{2}, 2\right),(1,1)$. Respectively we get these volumes: $\frac{121}{192}, \frac{25}{24}, \frac{2}{3}, \frac{7}{12}, \frac{1}{2}, 1$. The smallest is $\frac{1}{2}$ at $\left(\frac{1}{2}, 2\right)$ and the largest is $\frac{25}{24}$ at (1, $\frac{5}{4}$ ).
$\# 22$ Starting with $V=l w h$ and the constraint that $l+w+h \leq 45$. Since we want the largest volume, we'll want to actually hit the constraining value, so we'll use $l+w+h=45$. Solve for $h=45-l-w$, a. so in two variables $V=l w(45-l-w)=45 l w-l^{2}-l w^{2}$.
b. Surely $l$, $w, h \geq 0$. These constraints together give us, in particular $45-l-w \leq 0$, which closes the boundary. Altogether the vertices of the triangle are $(0,0),(0,45) \operatorname{and}(45,0)$. Each of them and the entire boundary has volume zero. It is not the maximum. It is the minimum.
c. $\nabla V=\left\langle 45 w-2 l w-w^{2}, 45 l-2 l w-l^{2}\right\rangle$. So, now we find where it is zero. We factor to get $w(45-2 l-w)=0$ and $l(45-2 w-l)=0$. We get $(w=0$ or $45-2 l-w=0)$ and $(l=0$ or $45-2 w-l=0)$. This produces four solutions: $(0,0),(0,45),(45,0)$, and $(15,15)$. The first three are corners.
d. The last one is the maximal point, with volume $15^{3}=3375$. Notice this is bigger than actual bags, but this cube-shaped bag probably would not fit into the overhead bins, and we're not given that information. This makes sense that the ones with the extra constraints would be smaller.
\#23 This is the canonical derivation of how we do linear regression. Start with $S(m, b)=\sum_{i=1}^{n}\left(m x_{i}+b-y_{i}\right)^{2}$. Remember, it's just a sum and the sum plays nicely with derivatives. Also remember that $x_{i}$ and $y_{i}$ are constants. The variables are $m$ and $b$.
a. Find $\frac{\partial S}{\partial m}=\sum_{i=1}^{n} 2\left(m x_{i}+b-y_{i}\right) x_{i}$ and $\frac{\partial S}{\partial b}=\sum_{i=1}^{n} 2\left(m x_{i}+b-y_{i}\right)$. Now that we have no squares here, let's distribute these sums. Because they are finite sums, everything commutes nicely:

$$
\begin{gathered}
\frac{\partial S}{\partial m}=\sum_{i=1}^{n} 2\left(m x_{i}+b-y_{i}\right) x_{i}=2\left(\sum_{i=1}^{n} m x_{i}^{2}+\sum_{i=1}^{n} b x_{i}-\sum_{i=1}^{n} y_{i} x_{i}\right)=2\left(m \sum_{i=1}^{n} x_{i}^{2}+b \sum_{i=1}^{n} x_{i}-\sum_{i=1}^{n} y_{i} x_{i}\right) \\
\frac{\partial S}{\partial b}=\sum_{i=1}^{n} 2\left(m x_{i}+b-y_{i}\right)=2\left(\sum_{i=1}^{n} m x_{i}+\sum_{i=1}^{n} b-\sum_{i=1}^{n} y_{i}\right)=2\left(m \sum_{i=1}^{n} x_{i}+n b-\sum_{i=1}^{n} y_{i}\right)
\end{gathered}
$$

b. Solve $\nabla S=0$. (First divide the twos away.) Let's solve the second equation for $b$ since it's pretty easy to do: $b=\frac{1}{n} \sum_{i=1}^{n} y_{i}-\frac{m}{n} \sum_{i=1}^{n} x_{i}$. So, we substitute this into the second:

$$
0=m \sum_{i=1}^{n} x_{i}^{2}+\left(\frac{1}{n} \sum_{i=1}^{n} y_{i}-\frac{m}{n} \sum_{i=1}^{n} x_{i}\right) \sum_{i=1}^{n} x_{i}-\sum_{i=1}^{n} y_{i} x_{i}
$$

Multiply all by $n$ and move the terms without $m$ to the left to get:

$$
n \sum_{i=1}^{n} y_{i} x_{i}-\sum_{i=1}^{n} y_{i} \sum_{i=1}^{n} x_{i}=m n \sum_{i=1}^{n} x_{i}^{2}-m \sum_{i=1}^{n} x_{i} \sum_{i=1}^{n} x_{i}
$$

Finally factor and solve for $m$ to yield:

$$
m=\frac{n \sum_{i=1}^{n} y_{i} x_{i}-\sum_{i=1}^{n} y_{i} \sum_{i=1}^{n} x_{i}}{n \sum_{i=1}^{n} x_{i}^{2}-\left(\sum_{i=1}^{n} x_{i}\right)^{2}}
$$

Now, we substitute $m$ back in to $b$. (Although this is not how this is done in practice. You just find $m$ first and don't have a formula for $b$ aside from the one that uses $m$. ... I don't feel like typesetting this when it is not what anyone would ever do, so I'm going to leave my formula for $b$ that I have.)
c. The second derivative sounds scary here, but I am hopeful. $\frac{\partial^{2} S}{\partial m^{2}}=2 \sum_{i=1}^{n} x_{i}^{2}, \frac{\partial^{2} S}{\partial b \partial m}=2 \sum_{i=1}^{n} x_{i}$, and $\frac{\partial^{2} S}{\partial b^{2}}=2 n$. So, $D=4 n \sum_{i=1}^{n} x_{i}^{2}-4\left(\sum_{i=1}^{n} x_{i}\right)^{2}$. Well, this looks positive. The first term includes $n$. Notice also that $\frac{\partial^{2} S}{\partial b^{2}}=2 n$ is surely positive, so this is a minimum if $D>0$. I'm willing to say it looks good. This is an absolute minimum because it is the only critical point and it is a minimum and there is no bound on the region.
d. I'm saying ... they didn't give me a scatter plot, so I'm not drawing a line. This is statistics. You can substitute into a formula or excel, or any as you wish. Tell me if you have a problem with this.
10.8 \#1 Create a contour plot for $f(x, y)=x^{2}+x y+y^{2}$. We will consider minimizing $f$ subject to the constraint $g(x, y)=x+y-2=0$. (There is no constrained maximum.)
a. Carefully draw the constraint set $g(x, y)=0$ into the picture. Label some contour lines with their $z$-values.

Maple: "contourplot $\left(x^{2}+x * y+y^{2}, x=-3 . .3, y=-3 . .3\right.$, axes=boxed )" The most important contour lines appears to be the inner oval which has $z=3$
b. Using the picture alone, estimate the points at which the constrained minimum occurs and the values of $f$ at this point.

There is one place where the constraint is tangent to a level curve, it appears to be (1, 1). And the value appears to be $f(1,1)=3$.
c. Use the Lagrange multiplier condition to check your work in the previous part.

So, $\nabla f(x, y)=\lambda \nabla g(x, y)$ i.e. $(2 x+y, x+2 y)=\lambda(1,1)$. We have three equations: $2 x+y=\lambda, x+2 y=\lambda$, and $x+y-2=0$. These are pretty easy equations to produce $(1,1)$ as claimed. The point here is to see the picture so you understand our method.
\#14 Practice Lagrange multipliers:
a. $f(x, y)=(x-1)^{2}+(y-2)^{2}, g(x)=x^{2}+y^{2}=16$. So, $\nabla f(x, y)=(2 x-2,2 y-4)=\lambda(2 x, 2 y)=\lambda \nabla g(x, y)$. This produces $x-1=\lambda x$ and $y-2=\lambda y$ along with $x^{2}+y^{2}=16$. The first yields $x=\frac{1}{1-\lambda}$ and the second $y=\frac{2}{1-\lambda}$. We could substitute these into the third equation, but notice $y=2 x$. This is easier. So, we have $x^{2}+4 x^{2}=16$, hence $x= \pm \frac{4}{\sqrt{5}}$ and $y= \pm \frac{8}{\sqrt{5}}$. (Notice we never found $\lambda$, but we also don't care about it anymore.) At these points we get $f(x, y)=21 \mp 8 \sqrt{5}$. So, the positive point produces a minimum, and the negative point produces a maximum. Note the goal function gives us the closest point to $(1,2)$ (the one in the first quadrant) and the furthest point from $(1,2)$ on the given circle.
b. Comparing to last time $f(x, y, z)=(x-3)^{2}+(y-1)^{2}+(z+1)^{2}, g(x, y, z)=x^{2}+y^{2}+z^{2}=4$. So, $\nabla f(x, y, z)=(2 x-6,2 y-2,2 z+2)=\lambda(2 x, 2 y, 2 z)=\lambda \nabla g(x, y, z)$. This produces $x-3=\lambda x, y-1=\lambda y$, and $z+1=\lambda z$, along with $x^{2}+y^{2}+z^{2}=4$. The first yields $x=\frac{3}{1-\lambda}$ and the second $y=\frac{1}{1-\lambda}$ with the third $z=-\frac{1}{1-\lambda}$. We could substitute these into the final equation, but notice $x=3 y$ and $z=-y$. Again this is easier, so we have $9 y^{2}+y^{2}+y^{2}=4$, hence $y= \pm \frac{2}{\sqrt{11}}$ and $x= \pm \frac{6}{\sqrt{11}}$ and $z=\mp \frac{2}{\sqrt{11}}$. We test and find $f(x, y, z)=15 \mp 4 \sqrt{11}$, so the closest point is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}},-\frac{2}{\sqrt{11}}\right)$ and the furthest is $\left(-\frac{6}{\sqrt{11}},-\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$. Notice again the close one is in the same octant, and the far one is in the opposite.
c. Here we're looking for the closest point to the origin inside the sphere of radius 4 centred at $(3,-2,5)$. Notice that the origin itself is not included. Let's worry about critical values first, since that will be easy. $f(x, y, z)=x^{2}+y^{2}+z^{2}$, and $\nabla f(x, y, z)=(2 x, 2 y, 2 z)$. This is only the zero vector at the origin itself, which is not in the constraint. (Notice, it if were it would surely be the minimal value). Ok, so now routine Lagrange multipliers. $g(x, y, z)=(x-3)^{2}+(y+2)^{2}+(z-5)^{2}=16$ (we did inside already nothing to see there). So, $\nabla f(x, y, z)=(2 x, 2 y, 2 z)=\lambda(2 x-6,2 y+4,2 z-10)=\lambda \nabla g(x, y, z)$. This produces $x=\lambda(x-3), y=\lambda(y+2)$, and $z=\lambda(z-5)$, along with $(x-3)^{2}+(y+2)^{2}+(z-5)^{2}=16$. The first yields $x=\frac{3}{1-1 / \lambda}$ and the second $y=-\frac{2}{1-\lambda}$ with the third $z=\frac{5}{1-1 / \lambda}$. We could substitute these into the final equation, but notice $x=3 t, y=-2 t$ and $z=5 t$. Again this is easier, so we have
$(3 t-3)^{2}+(-2 t+2)^{2}+(5 t-5)^{2}=38(t-1)^{2}=16$, hence $t=1 \pm \frac{4}{\sqrt{38}}, x=3 \pm \frac{12}{\sqrt{38}}, y=-2 \mp \frac{8}{\sqrt{38}}$ and $z=5 \pm \frac{20}{\sqrt{38}}$. We test and find $f(x, y, z)=\frac{136 \sqrt{38}+1046}{19}, 54-8 \sqrt{38}$, so the closest point is $\left(3-\frac{12}{\sqrt{38}}\right.$, $\left.-2+\frac{8}{\sqrt{38}}, 5-\frac{20}{\sqrt{38}}\right)$ and the furthest is $\left(3+\frac{12}{\sqrt{38}},-2-\frac{8}{\sqrt{38}}, 5+\frac{20}{\sqrt{38}}\right)$.

