223 Assignment 6 Solutions

11.5 #1 These two intersect at the unit circle centred at the origin. Note $z = r^2$ and $z = 2 - r^2$. What we really have here is a triple integral in cylindrical coordinates: $\int \int \int_R 1r dz dr d\theta = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r dz dr d\theta$ but I expect you started with

$$\int_{0}^{2\pi} \int_{0}^{1} (2 - 2r^{2}) r dr d\theta = \int_{0}^{2\pi} \int_{0}^{1} 2r - 2r^{3} dr d\theta = \int_{0}^{2\pi} \left[r^{2} - \frac{1}{2}r^{4} \Big|_{0}^{1} d\theta = \int_{0}^{2\pi} \frac{1}{2} d\theta = \pi$$

§11.5 #3 For all of these, please ask if you wish to see the regions. I will attempt to describe. You get to choose two; I must do all four. Alas.

a. The region is a quarter of a circle of radius 3 in the third quadrant. So, the integral in rectangular coordinates is $\int_{-3}^{0} (\int_{-\sqrt{9-x^2}}^{0} x^2 + y^2 dy) dx$ (notice that one of the rs in r^3 goes to be part of $r dr d\theta$). I hope you're not surprised that I would rather do the polar integral: $\int_{\pi}^{3\pi/2} \int_{0}^{3} r^{3} dr d\theta = \int_{\pi}^{3\pi/2} \left[\frac{r^{4}}{4} \right]_{0}^{3} d\theta = \int_{\pi}^{3\pi/2} \left[\frac{r^{4}}{4} \right]_{0}^{3} d\theta$ $\int_{\pi}^{3\pi/2} \frac{81}{4} d\theta = \frac{81\pi}{8}.$ b. This is the one I mentioned on 21 November - the region is a circle centred at (1,0) of radius 1.

Shifting coordinates actually doesn't help because it messes up the integrand. If you're lucky, you remember something from Calc II, in particular that the circle to the right is parametrised by $r = \cos \theta$, but that

something from Calc II, in particular that the circle to the right is parametrised by $r = \cos \theta$, but that has radius $= \frac{1}{2}$, so use $r = 2\cos\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So, we get $\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^2 dr d\theta$, where one of the rs comes from $rdrd\theta$. This integral looks not great, but surely better than the original, so away we go: $\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^2 dr d\theta = \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\theta}{3} d\theta = \int_{-\pi/2}^{\pi/2} \frac{8((1-\sin^2\theta)\cos\theta}{3} d\theta) = \frac{8}{3} \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = \frac{32}{9}$. c. Here is the opposite of b., $r = \sin\theta$ is a circle of radius $\frac{1}{2}$ centred at $(0, \frac{1}{2})$. This goes from 0 to $\frac{\pi}{2}$ so it produces a half circle in the first quadrant. So, we have $\int_{0}^{1} \int_{0}^{\sqrt{\frac{1}{4} - (y - \frac{1}{2})^2}} \sqrt{1 - x^2 - y^2} dx dy$. Again that rectangular integral looks rough. And the polar one has some promising signs, so we re-sume: $\int_{0}^{\pi/2} \int_{0}^{\sin\theta} r\sqrt{1 - r^2} dr d\theta = \int_{0}^{\pi/2} \left[\frac{(1 - r^2)^{3/2}}{3} \right]_{0}^{\sin\theta} d\theta = \int_{0}^{\pi/2} \frac{1}{3}\cos^3\theta = \int_{0}^{\pi/2} \frac{1}{3}(1 - \sin^2\theta)\cos\theta d\theta =$

 $\frac{1}{3} \left[\sin \theta - \frac{1}{3} \sin^3 \theta \right]_0^{\pi/2} = \frac{2}{9}.$ d. The region here is deceiving in rectangular but quiet nice in polar - it is the first eighth of a unit circle. So, our polar integral becomes $\int_0^{\pi/4} \int_0^1 \cos(r^2) r dr d\theta$. Again, I select the polar one. I think this is rigged. $\int_0^{\pi/4} \int_0^1 \cos(r^2) r dr d\theta = \int_0^{\pi/4} \left[\frac{1}{2} \sin(r^2) \right]_0^1 d\theta = \int_0^{\pi/4} \frac{1}{2} \sin(1) d\theta = \frac{\pi}{8} \sin(1)$.

§11.6 #2 Note, here we use "sensible" spherical coordinates, i.e. latitude and longitude, not the silliness from $\S11.8$. We derived:

$$x(s,t) = R\cos(s)\cos(t), y(s,t) = R\cos(s)\sin(t), z(s,t) = R\sin(s)$$

We can write this as $\mathbf{r}(s,t) = \langle R\cos(s)\cos(t), R\cos(s)\sin(t), R\sin(s) \rangle$ Now we can compute $\frac{\partial \mathbf{r}}{\partial s} = \langle -R\sin(s)\cos(t), -R\sin(s)\sin(t), R\cos(s) \rangle$ and $\frac{\partial \mathbf{r}}{\partial t} = \langle -R\cos(s)\sin(t), R\cos(s)\cos(t), 0 \rangle$. Now, we need $\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right| = \left|\langle -R^2\cos^2 s\cos t, -R^2\cos^2 s\sin t, -R^2\cos s\sin s\rangle\right| = \left|-R^2\cos s(\cos s\cos t, \cos s\sin t, \sin s)\right| = R^2\cos s(\sqrt{\cos^2 s\cos^2 t + \cos^2 s\sin^2 t + \sin^2 s}) = R^2\cos s$. Finally we can set up our integral: $\int_0^{\pi/2} \int_0^{\pi/2} R^2\cos sdsdt$ is for the all-positive octant. This is about the easiest integral we've seen this chapter: $\int_0^{\pi/2} \left(R^2 \sin s \Big|_0^{\pi/2} dt = 0 \right)^{\pi/2} dt$ $\int_0^{\pi/2} R^2 dt = \frac{R^2 \pi}{2}$. Because it's an *oct*ant we multiply by 8 to get the entire sphere which gives us the comforting result of $4\pi R^2$.

§11.7 #1 The region of integration here is clearly presented as a triangle underneath a cone. The range of z values is 0 to $3\sqrt{x^2+y^2}$. I might like to try cylindrical coordinates because of that, but I don't think the triangle will be well suited. I'm going to stick to rectangular until I run into a stumbling point. The range of y values is 0 to 2 - x, and the range of x values is 0 to 2.

$$\int_{0}^{2} \left(\int_{0}^{2-x} \left(\int_{0}^{3\sqrt{x^{2}+y^{2}}} xz dz \right) dy \right) dx = \int_{0}^{2} \left(\int_{0}^{2-x} \left[\frac{1}{2}xz^{2} \Big|_{0}^{3\sqrt{x^{2}+y^{2}}} dy \right) dx = \int_{0}^{2} \left(\int_{0}^{2-x} \frac{9}{2}(x^{3}+xy^{2}) dy \right) dx$$
$$= \int_{0}^{2} \left[\frac{9}{2}x^{3}y + \frac{3}{2}xy^{3} \Big|_{0}^{2-x} dx = \int_{0}^{2} 12x - 18x^{2} + 18x^{3} - 6x^{4} dx = \left[6x^{2} - 6x^{3} + \frac{9}{2}x^{4} - \frac{6}{5}x^{5} \Big|_{0}^{2} = \frac{48}{5} \right]_{0}^{2}$$

§11.7 #3 a. The two surfaces intersect at $4x^2 + 4y^2 = 8$, i.e. $x^2 + y^2 = 2$, a circle of radius $\sqrt{2}$ centred at the origin.

b. To find centre of mass, we integrate density. The xy-projection is the circle, and one surface is top and one bottom. We find $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{8-3x^2-3y^2} \delta dz dy dx$. I'm not sure what they meant by x first, since this seems x last to me, but it's the only natural way to set it up.

c. Yuck to doing this in other orders. One is easy: $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{x^2+y^2}^{8-3x^2-3y^2} \delta dz dx dy$. And luckily we always get two-for-one because of x - y symmetry. So, we must do the four others. Each of the other ways will require breaking into two integrals. Let's try the less-unpleasant one first: dydzdx. The range of x values remains the same. The z values are bounded above by $8 - 3x^2$ and below by x^2 , by finding the projection, which happens conveniently when y = 0. But, then the range of y values is a mess. It depends on the particular z value. For $z \leq 2$ we solve $z = x^2 + y^2$ for $y = \pm \sqrt{z - x^2}$, and for $z \ge 2$ we solve $z = 8 - 3x^2 - 3y^2$ for $y = \pm \sqrt{\frac{8 - 3x^2 - z}{3}}$. So putting this together we have $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2}^2 \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} \delta dy dz dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_2^{8-3x^2} \int_{-\sqrt{\frac{8-3x^2-z}{3}}}^{\sqrt{\frac{8-3x^2-z}{3}}} \delta dy dz dx.$ We do get two-for-one, so can switch x and $y \text{ to get } \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^2 \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} \delta dx dz dy + \int_{-\sqrt{2}}^{\sqrt{2}} \int_{2}^{8-3y^2} \int_{-\sqrt{\frac{8-3y^2-z}{3}}}^{\sqrt{\frac{8-3y^2-z}{3}}} \delta dx dz dy.$

Now, what about dydxdz? The range of z values is from 0 to 8, but will probably need to be broken at 2 again. The curve $z = x^2$ determines left-right bounds for the bottom to be between $\pm \sqrt{z}$, and for the top we have $\pm \sqrt{\frac{8-z}{3}}$. Then for y limits we solve $z = x^2 + y^2$ and $z = 8 - 3x^2 - 3y^2$ for y in each case to get $\pm \sqrt{z-x^2}$ and $\pm \sqrt{\frac{8-3x^2-z}{3}}$, as before. Assembling we produce

$$\int_{0}^{2} \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^{2}}}^{\sqrt{z-x^{2}}} \delta dy dx dz + \int_{2}^{8} \int_{-\sqrt{\frac{8-z}{3}}}^{\sqrt{\frac{8-z}{3}}} \int_{-\sqrt{\frac{8-3x^{2}-z}{3}}}^{\sqrt{\frac{8-3x^{2}-z}{3}}} \delta dy dx dz \text{ and correspondingly}$$

 $\int_{0}^{2} \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^{2}}}^{\sqrt{z-y^{2}}} \delta dx dy dz + \int_{2}^{8} \int_{-\frac{1}{3}\sqrt{8-z}}^{\frac{1}{3}\sqrt{8-z}} \int_{-\sqrt{\frac{8-3y^{2}-z}{3}}}^{\sqrt{\frac{8-3y^{2}-z}{3}}} \delta dx dy dz.$ Although, all of this is silly, why wouldn't we just use the first way? Alas. Fortunately the rest of this problem is easy now. d. This is easy from b. Compute $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^{2}}}^{\sqrt{2-x^{2}}} \int_{x^{2}+y^{2}}^{8-3x^{2}-3y^{2}} x \delta dz dy dx, \quad \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^{2}}}^{\sqrt{2-x^{2}}} \int_{x^{2}+y^{2}}^{8-3x^{2}-3y^{2}} \delta dz dy dx.$ and $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^{2}}}^{\sqrt{2-x^{2}}} \int_{x^{2}+y^{2}}^{8-3x^{2}-3y^{2}} z \delta dz dy dx, \text{ then divide each by } \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^{2}}}^{\sqrt{2-x^{2}}} \int_{x^{2}+y^{2}}^{8-3x^{2}-3y^{2}} \delta dz dy dx.$ This will tell us the three coordinates of centre of mass. Do *not* do this in the other orders in c., that would be too silly.

e. Notice that for this problem I have been ignoring δ the entire time. Now I need to notice one thing about it if we substitute -x in for x or -y in for y we get the same result. Because of this and because the solid is symmetric across the y = 0 and x = 0 planes, the x and y coordinates of centre of mass will both be 0. The z coordinate is a mystery, probably somewhere above 2, but that's not thinking about the density, which is higher for lower z. That would take the integral. Fortunately, we're not asked to do it.

§11.8 #1 This really should be cylindrical coordinates. I can't imagine doing it any other way. So, I will. The cone is z = r and the paraboloid is $z = 2 - r^2$. They equal when r = 1 (or -2, but that doesn't make sense for the square root). So, $\int \int \int_R f(x, y, z) dV = \int \int \int_R f(r, \theta, z) r dz dr d\theta$

$$\begin{split} &= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} (r\cos\theta + r\sin\theta + z) r dz dr d\theta = \int_0^{2\pi} \int_0^1 \left[zr^2\cos\theta + zr^2\sin\theta + \frac{1}{2}z^2r \Big|_r^{2-r^2} dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r - r^3 - r^2)(\cos\theta + \sin\theta) + \frac{1}{2}(r^5 - 5r^3 + 4r) dr d\theta \\ &= \int_0^{2\pi} \left[(r^2 - \frac{1}{4}r^4 - \frac{1}{3}r^3)(\cos\theta + \sin\theta) + \frac{1}{12}r^6 - \frac{5}{8}r^4 + 2r^2 \Big|_0^1 d\theta = \int_0^{2\pi} \frac{10\cos\theta + 10\sin\theta + 11}{24} d\theta \\ &= \left[\frac{10\sin\theta - 10\cos\theta + 11\theta}{24} \Big|_0^{2\pi} = \frac{11\pi}{12} \end{split}$$

 $\S11.8 \ \#2$ So, we are seeking the sphere ...

a. Let's decompose the limits: x goes from 0 to 1, and y from 0 to $\sqrt{1-x^2}$, which gives us $x^2 + y^2 = 1$, so the xy projection is a quarter circle in the first quadrant, so far so good. z ranges from $\sqrt{x^2 + y^2}$ to $\sqrt{2-x^2-y^2}$, notice the second gives us $x^2 + y^2 + z^2 = 2$ which is a sphere of radius $\sqrt{2}$. That's very hopeful. And the first limit gives us z = r, which happens when $\phi = \frac{\pi}{4}$. To be sure that we stay in the first quadrant over xy we limit θ between 0 and $\frac{\pi}{2}$. The distance from the origin can range from 0 to $\sqrt{2}$, given by that sphere boundary above. Altogether this gives us (after changing the integrand with dV and exchanging xy)

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho \sin \phi \cos \theta \rho \sin \phi \sin \theta \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^4 \sin^3 \phi \cos \theta \sin \theta d\rho d\phi d\theta$$

Notice this is a spherical box, so the integration is simplified because the variables do not cross reference. In particular because of this, we can separate them by pulling out constants:

$$= \left(\int_0^{\pi/2} \cos\theta \sin\theta d\theta\right) \left(\int_0^{\pi/4} \sin^3\phi d\phi\right) \left(\int_0^{\sqrt{2}} \rho^4 d\rho\right)$$

The last integral is easy. The first is a simple calc I substitution, and the middle is a standard calc 2 integral done by letting $\sin^3 \phi = (1 - \cos^2 \phi) \sin \phi$. Together they give:

$$= \left(\frac{1}{2}\right) \left(\frac{8-5\sqrt{2}}{12}\right) \left(\frac{4\sqrt{2}}{5}\right) = \frac{4\sqrt{2}-5}{15}$$

b. is already in spherical coordinates, so it can't be the one. So, that leaves c. I like the z = r lower

bound, which will correspond nicely with $\phi = \frac{\pi}{4}$. Here's the next key for this transition: remember $z = \rho \cos \phi$ so z = 1 becomes $\rho = \frac{1}{\cos \phi} = \sec \phi$. I think that will serve us nicely for the limits. For the integrand $\cos \theta$ doesn't change, but since $r^2 = x^2 + y^2$ and $\rho^2 = x^2 + y^2 + z^2$, we have $\rho^2 = r^2 + z^2$, and so $r^2 = \rho^2 - z^2 = \rho^2(1 - \cos^2 \phi) = \rho^2 \sin^2 \phi$, hence $r = \rho \sin \phi$.

 $\rho^{2} = x^{2} + y^{2} + z^{2}, \text{ we have } \rho^{2} = r^{2} + z^{2}, \text{ and so } r^{2} = \rho^{2} - z^{2} = \rho^{2}(1 - \cos^{2}\phi) = \rho^{2} \sin^{-}\phi, \text{ nence } r = \rho \sin \phi.$ So, let's see what we can do with the integrand. Remember $dxdydz = rdzdrd\theta = \rho^{2} \sin \phi d\rho d\theta d\phi$. So, taking this piece-by-piece $r \cos \theta r dz dr d\theta = \rho \sin \phi \cos \theta \rho^{2} \sin \phi d\rho d\theta d\phi = \rho^{3} \sin^{2}\phi \cos \theta d\rho d\theta d\phi$. So, now let's assemble this with the limits: $\int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{\sec \phi} \rho^{3} \sin^{2}\phi \cos \theta d\rho d\phi d\theta.$ And away we go with integration. First is easy, then takes some work: $= \int_{0}^{2\pi} \int_{0}^{\pi/4} \left[\frac{1}{4}\rho^{4} \sin^{2}\phi \cos \theta\right]_{0}^{\sec \phi} d\phi d\theta = \int_{0}^{2\pi} \int_{0}^{\pi/4} \frac{1}{4} \sec^{4}\phi \sin^{2}\phi \cos \theta d\rho d\phi d\theta.$ Now, let's rearrange just the new integrand before trying to integrate it: $\sec^{4}\phi \sin^{2}\phi = \frac{1-\cos^{2}\phi}{\cos^{4}\phi}$ $=\sec^4\phi - \sec^2\phi = \sec^2\phi(1 + \tan^2\phi) - \sec^2\phi = \sec^2\phi\tan^2\phi, \text{ finally some good news. So, our integral is now for a set of the se$ $\int_{0}^{2\pi} \int_{0}^{\pi/4} \frac{1}{4} \sec^{2} \phi \tan^{2} \phi \cos \theta d\rho d\phi d\theta = \int_{0}^{2\pi} \left[\frac{1}{12} \tan^{3} \phi \right]_{0}^{\pi/4} \cos \theta d\theta = \int_{0}^{2\pi} \frac{1}{12} \cos \theta d\theta = \left[\frac{1}{12} \sin \theta \right]_{0}^{2\pi} = 0, \text{ really??}$ All that for zero? Now that I see this end, I wish I had stayed in cylindrical all along, because I can see that $\int_0^{2\pi} \int_0^1 \int_r^1 r^2 \cos\theta dz dr d\theta = \left(\int_0^{2\pi} \cos\theta d\theta\right) \left(\int_0^1 \int_r^1 r^2 dz dr\right) = 0.$ Hm.

§11.9 #1 First just follow the lines We have a parallelogram to begin with. The corners are at $\left(-\frac{4}{5}, -\frac{1}{5}\right)$, $\left(\frac{2}{5}, -\frac{7}{5}\right)$, $\left(\frac{2}{5}, \frac{8}{5}\right)$. and $\left(\frac{8}{5}, \frac{2}{5}\right)$. This is intended to not be pleasant, but I'm pleased with the repeated $\frac{2}{5}$ which will only produce two rather than three integrals to do. The left one is from $x = -\frac{4}{5}$ to $\frac{2}{5}$ and the right is from $\frac{2}{5}$ to $\frac{8}{5}$. On the left top we have 3x - 2y = -2 and bottom we have x + y = -1. On the right top we have x + y = 2 and bottom 3x - 2y = 4. This produces these integrals:

$$\int_{-\frac{4}{5}}^{\frac{2}{5}} \left(\int_{-1-x}^{\frac{3x+2}{2}} (x+y) dy \right) dx + \int_{\frac{2}{5}}^{\frac{8}{5}} \left(\int_{\frac{3x-4}{2}}^{2-x} (x+y) dy \right) dx = \int_{-\frac{4}{5}}^{\frac{2}{5}} \left[xy + \frac{1}{2}y^2 \Big|_{-1-x}^{\frac{3x+2}{2}} dx + \int_{\frac{2}{5}}^{\frac{8}{5}} \left[xy + \frac{1}{2}y^2 \Big|_{\frac{3x-4}{2}}^{2-x} dx + \int_{\frac{2}{5}$$

That's a mess. I'm hopeful it will be much prettier with a change of variables. Let's see. The new region is just the rectangle $[-2, 4] \times [-1, 2]$. Solving for $x = \frac{u+2v}{5}$ and $y = \frac{3v-u}{5}$, so our integrand becomes $\frac{u+2v}{5} + \frac{3v-u}{5} = v$. $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{1}{5} & \frac{2}{3} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} = \frac{1}{5}$. So now our integral is the same as

$$\int_{-2}^{4} \left(\int_{-1}^{2} v \frac{1}{5} dv \right) du = \int_{-2}^{4} \left[\frac{1}{10} v^{2} \right]_{-1}^{2} du = \int_{-2}^{4} \frac{3}{10} du = \frac{9}{5}.$$

Definitely met my hopes of being much prettier.

11.9 #2 For better or worse, I did this in class. I don't feel very bad about this. To find the Jacobian we need to find lots of partial derivatives. Before, we remember the coordinate system:

$$x(\rho, \theta, \phi) = \rho \sin(\phi) \cos(\theta), y(\rho, \theta, \phi) = \rho \sin(\phi) \sin(\theta), z(\rho, \theta, \phi) = \rho \cos(\phi)$$

Let's organise this this way, first take derivatives with respect to ρ :

$$\frac{\partial x}{\partial \rho}(\rho, \theta, \phi) = \sin(\phi)\cos(\theta), \\ \frac{\partial y}{\partial \rho}(\rho, \theta, \phi) = \sin(\phi)\sin(\theta), \\ \frac{\partial z}{\partial \rho}(\rho, \theta, \phi) = \cos(\phi)$$

Now take derivatives with respect to θ :

$$\frac{\partial x}{\partial \theta}(\rho, \theta, \phi) = -\rho \sin(\phi) \sin(\theta), \\ \frac{\partial y}{\partial \theta}(\rho, \theta, \phi) = \rho \sin(\phi) \cos(\theta), \\ \frac{\partial z}{\partial \theta}(\rho, \theta, \phi) = 0$$

Finally take derivatives with respect to ϕ :

Now, let's put this into a matrix, and find the determinant, and then (unlike in class) remember to take absolute value.

$$\begin{vmatrix} \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi) \\ -\rho\sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & 0 \\ \rho\cos(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) & -\rho\sin(\phi) \end{vmatrix} = -\rho^2\sin(\phi)$$

Then notice, since ρ^2 is clearly positive, and $\sin \phi$ is positive for $0 \le \phi \le \pi$, that $|-\rho^2 \sin(\phi)| = \rho^2 \sin(\phi)$, which is the change in volume element we derived in §11.8.